Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Finite Horizon Case

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Abstract

We analyze a finite horizon, single product, periodic review model in which pricing and production/inventory decisions are made simultaneously. Demands in different periods are random variables that are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period and all shortages are backlogged. Ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. The objective is to find an inventory policy and a pricing strategy maximizing expected profit over the finite horizon. We show that when the demand model is additive, the profit-to-go functions are k-concave and hence an (s, S, p) policy is optimal. In such a policy, the period inventory is managed based on the classical (s, S) policy and price is determined based on the inventory position at the beginning of each period. For more general demand functions, i.e., multiplicative plus additive functions, we demonstrate that the profit-to-go function is not necessarily k-concave and an (s, S, p) policy is not necessarily optimal. We introduce a new concept, the symmetric k-concave functions and apply it to provide a characterization of the optimal policy.

1 Introduction

Traditional inventory models focus on effective replenishment strategies and typically assume that a commodity's price is exogenously determined. In recent years, however, a number of industries have used innovative pricing strategies to manage their inventory effectively. For example, techniques such as *revenue management* have been applied in the airlines, hotels, and rental car agencies–integrating price, inventory control, and quality of service (see Kimes [6]). In the retail industry, to name another example, dynamically pricing commodities can provide significant improvements in profitability, as shown by Gallego and van Ryzin [5].

These developments call for models that integrate inventory control and pricing strategies. Such models are clearly important not only in the retail industry, where price-dependent demand plays an important role, but also in manufacturing environments in which production/distribution decisions can be complemented with pricing strategies to improve the firm's bottom line.

To date, the literature has confined itself mainly to models with variable ordering costs but no fixed costs. Extending some of these models to include a fixed cost component is the

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focus of our paper. Specifically, we consider a finite horizon, periodic review, single product model with stochastic demand. Demands in different periods are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period, and all shortages are backlogged. The ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. Inventory holding and shortage costs are convex functions of the inventory level carried over from one period to the next. The objective is to find an inventory policy and pricing strategy maximizing expected profit over the finite horizon.

Our model is similar to the model analyzed by Federgruen and Heching [4] except that in their model the authors assume that ordering cost is *proportional* to the amount ordered and thus does not include a fixed cost component. They show that in this case a *base-stock list price* policy is optimal. That is, in each period the optimal policy is characterized by an order-up-to level, referred to as the base-stock level, and a price which depends on the initial inventory level at the beginning of the period. If the initial inventory level is below the base-stock level an order is placed to raise the inventory level to the base-stock level. Otherwise, no order is placed and a discount price is offered. This discount price is a non-increasing function of the initial inventory level.

Of course, many papers address the coordination of replenishment strategies and pricing policies, starting with the work of Whitin [11] who analyzed the celebrated newsvendor problem with price dependent demand. For a review, the reader is referred to Eliashberg and Steinberg [3], Petruzzi and Dada [7], Federgruen and Heching [4] or Chan, Simchi-Levi and Swann [2].

The paper by Thomas [10] considers a model similar to ours, namely, a periodic review, finite horizon model with a fixed ordering cost and stochastic, price-dependent demand. The paper postulates a simple policy, referred to by Thomas as (s, S, p), which can be described as follows. The inventory strategy is an (s, S) policy: If the inventory level at the beginning of period t is below the reorder point, s_t, an order is placed to raise the inventory level to the order-up-to level, S_t. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of the period. Thomas provides a counterexample which shows that with a "few prices" (i.e., when price is restricted to a discrete set) this policy may fail to be optimal. Thomas goes on to say:

> If all prices in an interval are under consideration, it is conjectured that an (s, S, p) policy is optimal under fairly general conditions.

We prove that for additive demand models, the (s, S, p) policy proposed by Thomas is indeed optimal. Unfortunately, for general demand models, Thomas's conjecture fails. Especially, the (s, S, p) policy is not necessarily optimal for multiplicative demand models. In the general demand case, we provide a characterization of the optimal policy by employing a new concept, the symmetric k-convexity.

The paper is organized as follows. In Section 2 we review the main assumptions of our model. We also introduce a new but equivalent definition of k-convex functions. In Section 3 we employ the new definition and characterize the optimal inventory and pricing policies for **additive demand functions**. We show that in this case *the policy proposed* *by Thomas* is indeed *optimal*. In Section 4 we analyze general demand functions which may be **non-additive**. We demonstrate that in this case the profit-to-go function is not necessarily k-concave and an (s, S, p) policy is not necessarily optimal. We introduce the concept of symmetric k-convex functions and apply it to provide a characterization of the optimal policy. In Section 5 we apply our results to the model with no fixed cost, that is, the model analyzed by Federgruen and Heching, [4]. A key assumption in their paper is that the demand function is a linear function of the price, see Lemma 1 in [4]. Interestingly, we show that our approach extends the results obtained in [4] to more general demand functions, including general additive and multiplicative demand functions. Finally, in Section 6 we discuss extensions and provide concluding remarks.

2 The Model

Consider a firm that has to make production and pricing decisions over a finite time horizon with T periods. Demands in different periods are independent of each other. For each period t, $t = 1, 2 \ldots, T$, let

d_t = demand in period t
p_t = selling price in period t
$\underline{p}_t, \bar{p}_t$ are lower and upper bounds on p_t, respectively.

Throughout this paper, we concentrate on demand functions of the following forms:

Assumption 1 *For $t = 1, 2, \ldots, T$, the demand function satisfies*

$$d_t = D_t(p, \epsilon_t) := \alpha_t D_t(p_t) + \beta_t, \tag{1}$$

where $\epsilon_t = (\alpha_t, \beta_t)$, and α_t, β_t are two random variables with $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$. The random perturbations, ϵ_t, are independent across time.

Observe that, by scaling and shifting, the assumptions $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$ can be made without loss of generality. A special case of this demand function, the **additive** demand function, is analyzed in Section 3. In this case, the demand function is of the form $d_t = D_t(p) + \beta_t$. This implies that only β_t is a random variable while $\alpha_t = 1$. In Section 4 we analyze the general demand functions (1). Observe that a special case of the model analyzed in Section 4 is a model with the **multiplicative** demand function. In this case, the demand function is of the form $d_t = \alpha_t D_t(p)$, where α_t is a random variable. Finally observe that special cases of the function $D_t(p)$ include $D_t(p) = b_t - a_t p$ $(a_t > 0, b_t > 0)$ in the additive case and $D_t(p) = a_t p^{-b_t}$ $(a_t > 0, b_t > 1)$ in the multiplicative case; both are common in the economics literature (see [7]).

We assume the following.

Assumption 2 *For all t, $t = 1, 2, \ldots, T$, the inverse function of D_t, denoted by D_t^{-1}, is continuous and strictly decreasing. Furthermore, the expected revenue*

$$R_t(d) := dD_t^{-1}(d)$$

is a concave function of expected demand d.

Let x_t be the inventory level at the beginning of period t, just before placing an order. Similarly, y_t is the inventory level at the beginning of period t after placing an order. Define

$$\delta(u) := \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The ordering cost function includes both a fixed cost and a variable cost and is calculated for every t, $t = 1, 2, \ldots, T$, as

$$k\delta(y_t - x_t) + c_t(y_t - x_t).$$

Note that while the variable cost function is time dependent, the fixed cost function, k, is time independent. In fact, as we observe at the end of the paper, all our results can be extended to situations in which the fixed cost is a *non-increasing function* of time.

Unsatisfied demand is backlogged. Let x be the inventory level carried over from period t to period $t+1$. Since we allow backlogging, x may be positive or negative. A cost $h_t(x)$ is incurred at the end of period t which represents inventory holding cost when $x > 0$ and penalty cost if $x < 0$. We denote the expected inventory holding and penalty cost by

$$G_t(y, p) = E\{h_t(y - D_t(p, \epsilon_t))\}.$$

For technical reasons, we need the following assumptions regarding properties of function $G_t(y, d)$ and the finiteness of the moments of the demand functions. These assumptions are similar to those in [4].

Assumption 3 *For each t, $t = 1, 2, \ldots, T$, $h_t(x)$, is a convex function of the inventory level x at the end of period t. Furthermore,* $\lim_{y\to\infty} G_t(y, D_t(p, \epsilon_t)) = \lim_{y\to-\infty}[c_t y + G_t(y, D_t(p, \epsilon_t))] = \lim_{y\to\infty}[(c_t - c_{t+1})y + G_t(y, D_t(p, \epsilon_t))] = \infty$ *for all* $p \in [\underline{d}_t, \bar{p}_t]$.

Assumption 4 $0 \leq G_t(y, D_t(p, \epsilon_t)) = O(|y|^\rho)$ *for some integer* ρ.

Assumption 5 $E\{D_t(p, \epsilon_t))\}^\rho < \infty$ *for all* $p \in [\underline{p}_t, \bar{p}_t]$.

The objective is to decide on ordering and pricing policies so as to maximize total expected profit over the entire planning horizon. That is, the objective is to choose y_t and p_t so as to maximize

$$E\{\sum_{t=1}^{T} -k\delta(y_t - x_t) - c_t(y_t - x_t) - h_t(x_{t+1}) + p_t D_t(p_t, \epsilon_t)\}, \tag{2}$$

where $x_{t+1} = y_t - D_t(p_t, \epsilon_t)$.

Denote by $v_t(x)$ the profit-to-go function at the beginning of time period t with inventory level x. A natural dynamic program for the above maximization problem is as follows. For $t = T, T-1, \ldots, 1$,

$$v_t(x) = c_t x + \max_{y \geq x, \bar{p}_t \geq p \geq \underline{p}_t} -k\delta(y - x) + f_t(y, p), \tag{3}$$

where

$$f_t(y,p) = -c_t y + E\{pD_t(p,\epsilon_t) - h_t(y - D_t(p,\epsilon_t)) + v_{t+1}(y - D_t(p,\epsilon_t))\}, \tag{4}$$

and $v_{T+1} = 0$. Let

$$p_t(y) \in \text{argmax}_{\bar{p}_t \geq p \geq \underline{p}_t}\ f_t(y,p). \tag{5}$$

Then

$$v_t(x) = c_t x + \max_{y \geq x} -k\delta(y-x) + f_t(y, p_t(y)).$$

For the general demand functions (1), we can present the formulation (4) only with respect to expected demand rather than with respect to price. Note that for $t = 1, 2, \ldots, T$, there is a one-to-one correspondence between the selling price $p_t \in [\underline{p}_t, \bar{p}_t]$ and the expected demand $D_t(p_t) \in [\underline{d}_t, \bar{d}_t]$, where

$$\underline{d}_t = D_t^{-1}(\bar{p}_t) \text{ and } \bar{d}_t = D_t^{-1}(\underline{p}_t).$$

We denote the expected demand at period t by $d = D_t(p)$. Then (4) can be rewritten as

$$g_t(y,d) = R_t(d) - c_t y + E\{-h_t(y - \alpha_t d - \beta_t) + v_{t+1}(y - \alpha_t d - \beta_t)\}, \tag{6}$$

the dynamic program (3) can be written as

$$v_t(x) = c_t x + \max_{y \geq x, \bar{d}_t \geq d \geq \underline{d}_t} -k\delta(y-x) + g_t(y,d), \tag{7}$$

and

$$f_t(y, p_t(y)) = g_t(y, d_t(y)) = \max_{\bar{d}_t \geq d \geq \underline{d}_t} g_t(y,d), \tag{8}$$

where $d_t(y) = D_t(p_t(y))$ is an optimal solution for the above maximization problem (8).

We now relate our problem to the celebrated stochastic inventory control problem discussed by Scarf [8]. In that problem demand is assumed to be exogenously determined, while in our problem demand depends on price. Other assumptions regarding the framework of the model are similar to those made by Scarf [8].

For the classical stochastic inventory problem Scarf [8] showed that an (s,S) policy is optimal. In this policy, the optimal decision in period t is characterized by two parameters, the reorder point, s_t, and the order-up-to level, S_t. An order of size $S_t - x_t$ is made at the beginning of period t if the initial inventory level at the beginning of the period, x_t, is smaller than s_t. Otherwise, no order is placed.

To prove that an (s,S) policy is optimal Scarf [8] uses the concept of k-convexity.

Definition 2.1 *A real-valued function f is called k-convex for $k \geq 0$, if for any $z \geq 0$, $b > 0$ and any y we have*

$$k + f(z+y) \geq f(y) + \frac{z}{b}(f(y) - f(y-b)). \tag{9}$$

A function f is called k-concave if $-f$ is k-convex.

For the purpose of the analysis of our model, we find it useful to introduce another, yet equivalent, definition of k-convexity.

Definition 2.2 *A real-valued function f is called k-convex for $k \geq 0$, if for any $x_0 \leq x_1$ and $\lambda \in [0, 1]$,*

$$f((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)f(x_0) + \lambda f(x_1) + \lambda k. \tag{10}$$

Proposition 1 *Definitions 2.1 and 2.2 are equivalent.*

Proof. For any $x_0 < x_1$, let

$$y = (1 - \lambda)x_0 + \lambda x_1, z = x_1 - y \text{ and } b = y - x_0, \tag{11}$$

then $\lambda = b/(b + z)$, and by simple algebra (10) can be rewritten as (9).

On the other hand, for any $z \geq 0, b > 0$ and y, let $\lambda = b/(b + z)$, $x_0 = y - b$ and $x_1 = y + z$, and by simple algebra we have that (9) can be rewritten as (10). ∎

Definition 2.2 emphasizes the difference between k-convexity and the traditional convexity (which is also 0-convexity). It is clear from this definition that one significant difference between k-convexity and traditional convexity is that (10) is not symmetric with respect to x_0 and x_1.

It turns out that this asymmetry is the main barrier when trying to identify the optimal policy to our problem for non-additive demand functions. Indeed, in Section 4 we provide counterexamples to show that the profit-to-go function is *not* necessarily k-concave and an (s, S, p) policy is not necessarily optimal. This motivates the development of a new concept, the *symmetric* k-concave function, which allows us to characterize the optimal policy in the general demand case.

However, under the additive demand model analyzed in Section 3 this concept is not needed. Indeed, we prove that, for additive demand functions, the profit-to-go function is k-concave and hence the optimal policy for problem (3) is an (s, S, p) policy. Formally, in this policy, every period, t, the inventory policy is characterized by two parameters, the reorder point, s_t, and the order-up-to level, S_t. An order of size $S_t - x_t$ is made at the beginning of period t if the initial inventory level at the beginning of the period, x_t, is smaller than s_t. Otherwise, no order is placed. The selling price in period t, p_t, is a function of the inventory level after an order was made and is determined by (5).

We summarize properties of k-convex functions as follows (see [1] for details).

Lemma 1 *(a) A real-valued convex function is also 0-convex and hence k-convex for all $k \geq 0$. A k_1-convex function is also a k_2-convex function for $k_1 \leq k_2$.*

(b) If $g_1(y)$ and $g_2(y)$ are k_1-convex and k_2-convex respectively, then for $\alpha, \beta \geq 0$, $\alpha g_1(y) + \beta g_2(y)$ is $(\alpha k_1 + \beta k_2)$-convex.

(c) If $g(y)$ is k-convex and w is a random variable, then $E\{g(y - w)\}$ is also k-convex, provided $E\{|g(y - w)|\} < \infty$ for all y.

(d) *If g is a continuous k-convex function and $g(y) \to \infty$ as $|y| \to \infty$, then there exists scalars s and S with $s \leq S$ such that*

  (i) *$g(S) \leq g(y)$, for all scalars y.*

  (ii) *$g(S) + k = g(s) < g(y)$, for all $y < s$.*

  (iii) *$g(y)$ is a decreasing function on $(-\infty, s)$.*

  (iv) *$g(y) \leq g(z) + k$ for all y, z with $s \leq y \leq z$.*

3 Additive Demand Function

In this section, we focus on additive demand functions, i.e., demand functions of the form

$$d_t = D_t(p_t) + \beta_t,$$

where β_t is a random variable. We make the following assumption.

Assumption 6 *The expected revenue $R_t(d)$ is a strictly concave function of expected demand d.*

Observe that a special case of the demand function is the *additive linear demand function* in which $d_t = b_t - a_t p_t + \beta_t$ with $b_t, a_t > 0$ for $t = 1, 2, \ldots, T$.

In the following, we show, by induction, that $g_t(y, d_t(y))$ is a k-concave function of y and $v_t(x)$ is a k-concave function of x. Therefore, the optimality of an (s, S, p) policy follows directly from Lemma 1.

To prove that v_t is k-concave we need the following lemma.

Lemma 2 *Suppose there exists a finite value $d_t(y)$ which maximizes (8) for any y. Then, $y - d_t(y)$ is a non-decreasing function of y.*

Proof. By contradiction. Assume that $y' > y$, while $y' - d_t(y') < y - d_t(y)$. Let

$$d = d_t(y') - (y' - y) \text{ and } d' = d_t(y) + (y' - y).$$

Since $d_t(y) < d, d' < d_t(y')$, d, d' are feasible for (8), and $d_t(y), d_t(y')$ are the optimal solutions of (8) with parameters y and y', respectively, we have that

$$g_t(y, d_t(y)) \geq g_t(y, d) \text{ and } g_t(y', d_t(y')) \geq g_t(y', d'),$$

Adding the two inequalities and using the definition of $g_t(y, d)$ in equation (6) with $\alpha_t = 1$, we have

$$R_t(d_t(y)) + R_t(d_t(y')) \geq R_t(d) + R_t(d'). \tag{12}$$

This is true since by definition $y - d_t(y) = y' - d'$ and $y' - d_t(y') = y - d$.

Since $d_t(y) < d, d' < d_t(y')$ and $d + d' = d_t(y) + d_t(y')$, there exist $\lambda, \mu \in (0, 1)$, such that $d = (1-\lambda)d_t(y) + \lambda d_t(y')$, $d' = (1-\mu)d_t(y) + \mu d_t(y')$ and $\lambda + \mu = 1$. From the strict concavity of $R_t(d)$, we know that

$$\begin{aligned} R_t(d) + R_t(d') &> (1-\lambda)R_t(d_t(y)) + \lambda R_t(d_t(y')) \\ &+ (1-\mu)R_t(d_t(y)) + \mu R_t(d_t(y')) \\ &= R_t(d_t(y)) + R_t(d_t(y')), \end{aligned}$$

which is a contradiction to (12). Therefore, the lemma holds. ■

The lemma thus implies that the higher the inventory level at the beginning of time period t, y_t, the higher the expected inventory level at the end of period t, $y_t - d_t(y_t)$. We are now ready to prove our main results for the additive demand model.

Theorem 3.1 *(a) For $t = T, T-1, \ldots, 1$, $f_t(y,p) = O(|y|^\rho)$ and $v_t(x) = O(|x|^\rho)$.*

(b) For $t = T, T-1, \ldots, 1$, $f_t(y,p)$ is continuous in (y,p) and $\lim_{|y|\to\infty} f_t(y,p) = -\infty$ for any $p \in [\underline{p}_t, \bar{p}_t]$. Hence for any fixed y, $f_t(y,p)$ has a finite maximizer, denoted by $p_t(y)$.

(c) For any $t = T, T-1, \ldots, 1$, $f_t(y, p_t(y)) (= g_t(y, d_t(y)))$ and $v_t(x)$ are k-concave.

(d) For $t = T, T-1, \ldots, 1$, there exist s_t and S_t with $s_t \le S_t$ such that it is optimal to order $S_t - x_t$ and set the selling price $p_t = p_t(S_t)$ when $x_t < s_t$, and not to order anything and set $p_t = p_t(x_t)$ when $x_t \ge s_t$.

Proof. By induction. The proof of part (a) follows from Assumptions 3, 4 and 5 and is similar to that of Theorem 1 in [4]. So we omit its proof.

Assume parts (a),(b),(c) and (d) holds for $t+1$. The continuity of $f_t(y,p)$ on (y,p) is easy to check. From part (d),

$$v_{t+1}(x) = \begin{cases} -k + g_{t+1}(S_{t+1}, d_{t+1}(S_{t+1})) + c_{t+1}x, & \text{if } x \le s_{t+1} \\ g_{t+1}(x, d_{t+1}(x)) + c_{t+1}x, & \text{if } x \ge s_{t+1}. \end{cases}$$

This equation implies that $E\{v_{t+1}(y - d_t - \beta_t) - c_{t+1}(y - d_t - \beta_t)\} \le v_{t+1}(S_{t+1}) - c_{t+1}S_{t+1}$ and since $G_t(y,p) = G_t(y, D_t^{-1}(d))$ is jointly convex in (y,d), we have that $\lim_{|y|\to\infty} f_t(y,p) = -\infty$ for any $p \in [\underline{p}_t, \bar{p}_t]$ uniformly by Assumption 3. Hence for any fixed y, $f_t(y,p)$ has a finite maximizer $p_t(y)$. Thus part (b) holds for period t.

We now focus on part (c). Note that since $g_t(y, d_t(y)) = f_t(y, p_t(y))$, we only need to show that $g_t(y, d_t(y)$ and $v_t(x)$ are k-concave based on the assumption that $v_{t+1}(x)$ is k-concave.

For any $y < y'$, and $\lambda \in [0,1]$, we have by Lemma 2 and the assumption that v_{t+1} is k-concave that

$$\begin{aligned} &v_{t+1}((1-\lambda)(y - d_t(y) - \beta_t) + \lambda(y' - d_t(y') - \beta_t)) \\ &\ge (1-\lambda)v_{t+1}(y - d_t(y) - \beta_t) + \lambda v_{t+1}(y' - d_t(y') - \beta_t) - \lambda k. \end{aligned}$$

In addition, the concavity of $R_t(d)$ implies that

$$R_t((1-\lambda)d_t(y)+\lambda d_t(y')) \geq (1-\lambda)R_t(d_t(y))+\lambda R_t(d_t(y')).$$

Since $h_t(x)$ is convex we also have

$$\begin{aligned}&-h_t((1-\lambda)(y-d_t(y)-\beta_t)+\lambda(y'-d_t(y')-\beta_t))\\ &\geq -(1-\lambda)h_t(y-d_t(y)-\beta_t)-\lambda h_t(y'-d_t(y')-\beta_t).\end{aligned}$$

Adding the last three inequalities and taking expectation we get

$$g_t((1-\lambda)y+\lambda y',(1-\lambda)d_t(y)+\lambda d_t(y')) \geq (1-\lambda)g_t(y,d_t(y))+\lambda g_t(y',d_t(y'))-\lambda k.$$

Since $d_t((1-\lambda)y+\lambda y')$ is the optimal d for (8) with the parameter y replaced by $(1-\lambda)y+\lambda y'$, we have

$$g_t((1-\lambda)y+\lambda y',d_t((1-\lambda)y+\lambda y')) \geq g_t((1-\lambda)y+\lambda y',(1-\lambda)d_t(y)+\lambda d_t(y')),$$

and hence,

$$g_t((1-\lambda)y+\lambda y',d_t((1-\lambda)y+\lambda y')) \geq (1-\lambda)g_t(y,d_t(y))+\lambda g_t(y',d_t(y'))-\lambda k,$$

that is, $g_t(y,d_t(y))$ is a k-concave function of y.

Since $D_t(p)$ is continuous in p and $f_t(y,p)$ is continuous in (y,p), $g_t(y,d)$ is continuous in (y,d) and $g_t(y,d_t(y))$ is continuous in y. Furthermore, since $\lim_{|y|\to\infty} f_t(y,p) = -\infty$ for any $p \in [\underline{p}_t, \bar{p}_t]$ uniformly (see the beginning of the proof), we have that $g_t(y,d_t(y)) \to -\infty$, as $|y| \to \infty$.

Thus, using Lemma 1 part (d) we have from the $k-$concavity of $g_t(y,d_t(y))$ that there exists $s_t < S_t$, such that S_t maximizes $g_t(y,d_t(y))$ and s_t is the smallest value of y for which $g_t(S_t,d_t(S_t)) = g_t(y,d_t(y))+k$, and

$$v_t(x) = \begin{cases} -k+g_t(S_t,d_t(S_t))+c_t x, & \text{if } x \leq s_t \\ g_t(x,d_t(x))+c_t x, & \text{if } x \geq s_t. \end{cases}$$

The k-concavity of v_t can be checked directly from the k-concavity of $g_t(y,d_t(y))$, see [1] for a proof.

Part (d) follows directly from part (c) and Lemma 1. ■

Remark 1 *The results of Lemma 2 and therefore Theorem 3.1 hold even when the expected revenue $R_t(d)$ does not satisfy Assumption assumpD, i.e., $R_t(d)$ is assumed to be only concave. In this case, it is possible that the optimization problem, (8), has multiple optimal solutions. The proof of Lemma 2 can be modified to show that the result holds if $d_t(y)$ is chosen to be the smallest value among all optimal solutions.*

An interesting question is whether $p_t(y)$ is a non-increasing function of y, as is the case for a similar model with no fixed cost (see [4]). Unfortunately, this property does not hold for our model.

Proposition 2 *The optimal price, $p_t(y)$ is not necessarily a non-increasing function of y.*

The proof is provided in Appendix A.

4 General Demand Functions

In this section, we focus on general demand functions (1). Observe that the additive demand function analyzed in the previous section is a special case of the general demand function (1). More importantly, multiplicative demand functions of the form $d_t = \alpha_t D_t(p)$ where $D_t(p) = a_t p^{-b_t}$ $(a_t > 0, b_t > 1)$, or demand functions of the form $D_t(p, \epsilon_t) = \beta_t + \alpha_t(b_t - a_t p)$ $(a_t > 0, b_t > 0)$, are also special cases.

Our objective in this section is two-fold. First, we demonstrate that under demand functions (1), $v_t(x)$ may not be k-concave and an (s, S, p) policy may fail to be optimal for problem (3). Second, we characterize the structure of the optimal policy for the general demand functions (1).

To characterize the optimal policy for the demand functions (1), one might consider using the same approach applied in Section 3. Unfortunately, in this case, the function $y - \alpha_t d_t(y)$ is not necessarily a non-decreasing function of y for all possible α_t, as is the case for additive demand functions. Hence, the approach employed in Section 3 does not work in this case.

Specifically, the next lemma, whose proof is given in Appendix B, illustrates that the profit-to-go function is in general not k-concave.

Lemma 3 *There exists an instance of problem (3) with a multiplicative demand function and time independent parameters such that the functions $g_{T-1}(y, d_{T-1}(y))$ and $v_{T-1}(x)$ are not k-concave.*

Of course, it is entirely possible that even if the functions $g_t(y, d_t(y))$ and $v_t(x)$ are not k-concave for some period t, the optimal policy is still an (s, S, p) policy. The next lemma, whose proof is given in Appendix C, shows that this is not true in general.

Lemma 4 *There exists an instance of problem (3) with multiplicative demand functions where an (s, S, p) policy is not optimal.*

To overcome these difficulties, we propose a weaker definition of k-convexity, referred to as symmetric k-convexity:

Definition 4.1 *A real-valued function f is called sym-k-convex for $k \geq 0$, if for any x_0, x_1 and $\lambda \in [0, 1]$,*

$$f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)f(x_0) + \lambda f(x_1) + \max\{\lambda, 1-\lambda\}k. \tag{13}$$

A function f is called sym-k-concave if $-f$ is sym-k-convex.

Observe that k-convexity is a special case of sym-k-convexity. The following lemma describes properties of sym-k-convex functions, properties that are parallel to those proven in Lemma 1.

Lemma 5 *(a) A real-valued convex function is also sym-0-convex and hence sym-k-convex for all $k \geq 0$. A sym-k_1-convex function is also a sym-k_2-convex function for $k_1 \leq k_2$.*

(b) If $g_1(y)$ and $g_2(y)$ are sym-k_1-convex and sym-k_2-convex respectively, then for $\alpha, \beta \geq 0$, $\alpha g_1(y) + \beta g_2(y)$ is sym-$(\alpha k_1 + \beta k_2)$-convex.

(c) If $g(y)$ is sym-k-convex and w is a random variable, then $E\{g(y-w)\}$ is also sym-k-convex, provided $E\{|g(y-w)|\} < \infty$ for all y.

(d) If g is a continuous sym-k-convex function and $g(y) \to \infty$ as $|y| \to \infty$, then there exists scalars s and S with $s \leq S$ such that

(i) $g(S) \leq g(y)$ for all y.

(ii) s is the smallest value x such that $g(x) = g(S) + k$. Therefore, $g(y) > g(s)$ for all $y < s$.

(iii) $g(y) \leq g(z) + k$ for all y, z with $(s+S)/2 \leq y \leq z$.

Proof. Parts (a),(b) and (c) follow directly from the definition of symmetric k-convexity. Hence we focus on part (d). Since g is continuous and $g(y) \to \infty$ as $|y| \to \infty$, there exists x and S with $x \leq S$ such that $g(S) \leq g(y)$ for all y and $g(x) = g(S) + k$. Let $s = \min\{x : g(x) = g(S) + k\}$.

To prove part (d)(iii) we consider two cases. First, for any $S \leq y \leq z$, there exists $\lambda \in [0,1]$ such that $y = (1-\lambda)S + \lambda z$, and we have from the definition of sym-k-convex that

$$g(y) \leq (1-\lambda)g(S) + \lambda g(z) + \max\{\lambda, 1-\lambda\}k \leq g(z) + k,$$

where the second inequality follows from the fact that S minimizes $g(x)$ implying that $g(S) \leq g(z)$, and because $\max\{\lambda, 1-\lambda\} \leq 1$.

In the second case, consider y such that $S \geq y \geq (s+S)/2$. In this case, there exists $1 \geq \lambda \geq 1/2$ such that $y = (1-\lambda)s + \lambda S$ and from the definition of sym-k-convex we have that

$$g(y) \leq (1-\lambda)g(s) + \lambda g(S) + \lambda k \leq g(S) + k \leq g(z) + k,$$

since $g(s) = g(S) + k$. Hence (i)-(iii) hold. ■

We are ready to show the main result for the general demand model. In the following, we show, by induction, that $g_t(y, d_t(y))$ is a sym-k-concave function of y and $v_t(x)$ is a sym-k-concave function of x. Hence a characterization of the optimal pricing and ordering policies follows from Lemma 5.

Theorem 4.1 *(a) For $t = T, T-1, \ldots, 1$, $f_t(y,p) = O(|y|^\rho)$ and $v_t(x) = O(|x|^\rho)$.*

(b) For $t = T, T-1, \ldots, 1$, $f_t(y,p)$ is continuous in (y,p) and $\lim_{|y|\to\infty} f_t(y,p) = -\infty$ for any $p \in [\underline{p}_t, \bar{p}_t]$. Hence for any fixed y, $f_t(y,p)$ has a finite maximizer, denoted by $p_t(y)$.

(c) For any $t = T, T-1, \ldots, 1$, $f_t(y, p_t(y)) (= g_t(y, d_t(y)))$ and $v_t(x)$ are sym-k-concave.

(d) For $t = T, T-1, \ldots, 1$, there exists s_t and S_t with $s_t \leq S_t$ such that it is optimal to order $S_t - x_t$ and set $p_t = p_t(S_t)$ when $x_t < s_t$, and not to order anything and set $p_t = p_t(x_t)$ when $x_t \geq (s_t + S_t)/2$.

Proof. The proof of parts (a) and (b) is similar to the proof of the same parts in Theorem 3.1. We now focus on part (c).

By induction. $v_{T+1}(x) = 0$ is sym-k-concave. From the sym-k-concavity of $v_{t+1}(x)$, we have that for any y, y',

$$\begin{aligned}
& v_{t+1}((1-\lambda)y + \lambda y' - \alpha_{t+1}((1-\lambda)d_{t+1}(y) + \lambda d_{t+1}(y')) - \beta_{t+1}) \\
&= v_{t+1}((1-\lambda)(y - \alpha_{t+1}d_{t+1}(y) - \beta_{t+1}) + \lambda(y' - \alpha_{t+1}d_{t+1}(y') - \beta_{t+1})) \\
&\geq (1-\lambda)v_{t+1}(y - \alpha_{t+1}d_{t+1}(y) - \beta_{t+1}) + \lambda v_{t+1}(y' - \alpha_{t+1}d_{t+1}(y') - \beta_{t+1}) - \max\{1-\lambda, \lambda\}k.
\end{aligned}$$

Also, we have that $E\{h_t(y - \alpha_t d - \beta_t)\}$ is jointly convex in (y, d) and $R_t(d)$ is concave by Assumption 2. Hence, the function $g_t(y, d_t(y))$, and therefore $f_t(y, p_t(y))$, are sym-k-concave.

Denote by $v_t^*(x) := v_t(x) - c_t x$. From Lemma 5 we have

$$v_t^*(x) = \begin{cases} -k + g_t(S_t, d_t(S_t)), & \text{if } x \in I_t \\ g_t(x, d_t(x)), & \text{if } x \notin I_t, \end{cases}$$

where S_t is the maximizer of $g_t(y, d_t(y))$ and $I_t = \{y \leq S_t : g_t(y, d_t(y)) \leq g_t(S_t, d_t(S_t)) - k\}$. Furthermore, $v_t^*(x) \geq g_t(x, d_t(x))$ for any x and $v_t^*(x) \geq -k + g_t(S_t, d_t(S_t))$ for any $x \leq S_t$.

Let s_t be defined as the smallest value of y for which $g_t(S_t, d_t(S_t)) = g_t(y, d_t(y)) + k$. Note that from Lemma 5, $(-\infty, s_t] \subset I_t$ and $[(s_t + S_t)/2, \infty) \subset I_t^c$, the complement of I_t.

We now prove that $v_t(x)$ is sym-k-concave. It suffices to prove that $v_t^*(x) := v_t(x) - c_t x$ is sym-k-concave, since $c_t x$ is linear in x. For any $x_0 \leq x_1$ and $\lambda \in [0, 1]$, denote by $x_\lambda = (1-\lambda)x_0 + \lambda x_1$.

We consider four different cases

<u>Case 1:</u> If $x_0, x_1 \notin I_t$, then $v_t^*(x_\mu) = g_t(x_\mu, d_t(x_\mu))$ for $\mu = 0, 1$ and $v_t^*(x) \geq g_t(x, d_t(x))$ for any x implying that

$$\begin{aligned}
v_t^*(x_\lambda) &\geq g_t(x_\lambda, d_t(x_\lambda)) \\
&\geq (1-\lambda)g_t(x_0, d_t(x_0)) + \lambda g_t(x_1, d_t(x_1)) - \max\{\lambda, 1-\lambda\}k \\
&\geq (1-\lambda)v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where the second inequality holds since $g_t(y, d_t(y))$ is sym-k-concave.

<u>Case 2:</u> If $x_1 \in I_t$, then $x_\lambda \leq S_t$ since $x_0 \leq x_1 \leq S_t$ and therefore

$$\begin{aligned}
v_t^*(x_\lambda) &\geq -k + g_t(S_t, d_t(S_t)) \\
&\geq (1-\lambda)g_t(S_t, d_t(S_t)) + \lambda(k + g_t(x_1, d_t(x_1))) - k \\
&\geq (1-\lambda)v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where the second inequality holds since $x_1 \in I_t$.

<u>Case 3:</u> If $x_1 \notin I_t, x_0 \in I_t$ and $x_\lambda \leq S_t$, we have

$$\begin{aligned}
v_t^*(x_\lambda) &\geq -k + g_t(S_t, d_t(S_t)) \\
&\geq (1-\lambda)(k + g_t(x_0, d_t(x_0))) + \lambda g_t(x_1, d_t(x_1)) - k \\
&\geq (1-\lambda)v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where the second inequality holds since $x_0 \in I_t$.

Case 4: If $x_1 \notin I_t, x_0 \in I_t$ and $x_\lambda \geq S_t$, there exists $0 \leq \mu \leq \lambda$, such that $x_\lambda = (1-\mu)S_t + \mu x_1$, and

$$\begin{aligned}
v_t^*(x_\lambda) &= g_t(x_\lambda, d_t(x_\lambda)) \\
&\geq (1-\mu)g_t(S_t, d_t(S_t)) + \mu g_t(x_1, d_t(x_1)) - \max\{\mu, 1-\mu\}k \\
&\geq (1-\lambda)g_t(S_t, d_t(S_t)) + \lambda g_t(x_1, d_t(x_1)) \\
&+ (\lambda-\mu)(g_t(S_t, d_t(S_t)) - g_t(x_1, d_t(x_1))) - k \\
&\geq (1-\lambda)(-k + g_t(S_t, d_t(S_t))) + \lambda g_t(x_1, d_t(x_1)) - \max\{\lambda, 1-\lambda\}k \\
&\geq (1-\lambda)v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k,
\end{aligned}$$

where the first inequality follows from the definition of sym-k-concavity of $g_t(y, d_t(y))$, the third inequality from the fact that $\mu \leq \lambda$ and S_t maximizes $g_t(y, d_t(y))$.

Therefore, $v_t(x)$ is sym-k-concave. ■

Theorem 4.1 thus implies that the optimal policy for problem (3) is characterized by two parameters s_t and S_t. When the inventory level x_t at the beginning of the period t is less than s_t, an order of size $S_t - x_t$ is made. If x_t is greater than $(s_t + S_t)/2$, no order is placed. However, it is possible that an order will be placed when the inventory level $x_t \in [s_t, (s_t + S_t)/2]$, depending on the problem instance. In any case, if an order is placed, it is to raise the inventory level to S_t.

5 Special Case: Zero Fixed-Cost

Federgruen and Heching ([4] analyzed the zero fixed cost model both in the finite horizon and infinite horizon cases. Focusing on the finite horizon model, a key assumption in their paper implied by their Lemma 1 is that the demand function, D_t is a linear function of the price. In fact, it is not clear at all that any other demand function satisfies their main assumption Assumption 5.

We now apply our results to the zero fixed cost case.

Corollary 1 *Consider our model with zero fixed-cost and general demand functions (1). In this case, a base stock list price policy is optimal.*

Proof. By Theorem 4.1, the functions v_t and $g_t(y, d_t(y))$, $t = 1, 2, \ldots, T$, are symmetric 0-concave and hence, from Definition 4.1, they are concave. The optimality of the base stock inventory policy directly follows from the concavity of $g_t(y, d_t(y))$ for $t = 1, 2, \ldots, T$.

We now show that there exists $d_t(y)$ which is non-decreasing and therefore the optimal price $p_t(y)$ is non-increasing. If R_t is strictly concave, the optimization problem $\max_{\bar{d}_t \geq d \geq \underline{d}_t} g_t(y, d)$ has unique optimal solution. However, when R_t is concave, it is possible that the optimization problem has multiple optimal solutions. In the latter case, we let

$$d_t(y) = \min\{\text{argmax}_{\bar{d}_t \geq d \geq \underline{d}_t} g_t(y, d)\}.$$

Assume that there exist $y < y'$ such that $d_t(y) > d_t(y')$. We have

$$g_t(y, d_t(y)) > g_t(y, d_t(y')) \text{ and } g_t(y', d_t(y')) \geq g_t(y', d_t(y)). \tag{14}$$

Adding the two inequalities and using the definition of $g_t(y,d)$ in equation (6), we have upon denoting $r(x) = -h_t(x) + v_{t+1}(x)$,

$$E\{r(y' - \alpha_t d_t(y') - \beta_t) - r(y' - \alpha_t d_t(y) - \beta_t)\} > E\{r(y - \alpha_t d_t(y') - \beta_t) - r(y - \alpha_t d_t(y) - \beta_t)\}$$

which cannot be true since r is concave and hence has non-increasing difference. Therefore, $d_t(y)$ is non-decreasing and $p_t(y)$ is non-increasing. ■

6 Extensions and Concluding Remarks

In this section we summarize our main results and report on some important extensions of the model and results.

We showed that when the demand model is additive, the profit-to-go functions are k-concave and hence an (s, S, p) policy is optimal. For more general demand functions, i.e., multiplicative plus additive functions, we demonstrated that the profit-to-go function is not necessarily k-concave and an (s, S, p) policy is not necessarily optimal. We introduced a new concept, the symmetric k-concave functions and applied it to provide a characterization of the optimal policy.

Some extensions of our model are in order:

- **Non-increasing Fixed Cost:** The analysis so far assumes time-independent fixed cost function. In fact, Lemma 1 part (a) and Lemma 5 part (a) imply that our results can be carried over to non-increasing fixed cost functions.

- **Infinite Time Horizon:** The analysis of the infinite time horizon is significantly more complex but the main results remain the same. This analysis is presented in a companion paper.

- **Markovian Demand Model:** The results obtained in this paper can be extended to a Markovian demand model where the demand distribution at every time period is determined by an exogenous Markov chain. Specifically, our results hold under assumptions similar to those employed by Sethi and Cheng (see [9]) on state dependent holding costs as well as fixed and variable ordering costs.

- **Markdown Model:** In this case we assume that price in period t, p_t, is constrained by $p_t \leq p_{t-1}$ for $t = 2, 3, \ldots, T$. In this case, the dynamic program (7) must be modified and it can be written as

$$\begin{aligned} v_t(x, d') &= c_t x + \max_{y \geq x, \max\{\underline{d}_t, d'\} \leq d \leq \bar{d}_t} -k\delta(y - x) + R_t(d) - c_t y \\ &\quad + E\{-h_t(y - \alpha_t d - \beta_t) + v_{t+1}(y - \alpha_t d - \beta_t, d)\}. \end{aligned}$$

  It turns out that Theorem 4.1 holds for the modified function $v_t(x, d')$ and hence the policy introduced in Section 4 is optimal under the markdown setting. This is true since the sym-k-convexity property can be easily extended to multi-variable functions.

Finally, it is appropriate to point out an important limitation of the model analyzed in this paper, namely the lack of capacity constraints. Indeed, a model with capacity limitations on the amount ordered is significantly more difficult to analyze than the model considered in this paper. Nevertheless, it is encouraging to discover that stochastic inventory/pricing models that incorporate fixed and variable costs can be the subject of rigorous analysis. For these problems, our analysis illustrates the limitation of the concept of k-convexity and the need for a different concept, the symmetric k-convexity concept.

7 Appendix A

Proof of Proposition 2: The following example shows that for additive demand functions, the optimal price $p_t(y)$ is not necessarily non-increasing.

Example: Consider the last two time periods of problem (3). Let

$$\begin{aligned}&k = 1, c_T = 0, h_T(x) = |x|, d_T = 4 - p, \bar{p}_T = \underline{p}_T = 1,\\ &c_{T-1} = 0, h_{T-1}(x) = \max\{0, -x\} + \tfrac{1}{2}\max\{0, x\}, d_{T-1} = 1 - p,\\ &\bar{p}_{T-1} = 1, \underline{p}_{T-1} = 0.\end{aligned}$$

Then

$$v_T(x) = \begin{cases} 3 - |x - 3|, & \text{for } x \geq 2, \\ 2, & \text{otherwise} , \end{cases}$$

and

$$f_{T-1}(y, p) = p(1 - p) + \begin{cases} 2 + (y - 1 + p), & \text{for } y - 1 + p \leq 0, \\ 2 - \frac{1}{2}(y - 1 + p), & \text{for } y - 1 + p \in [0, 2], \\ \frac{1}{2}(y - 1 + p), & \text{for } y - 1 + p \in [2, 3], \\ 6 - \frac{3}{2}(y - 1 + p), & \text{otherwise} . \end{cases}$$

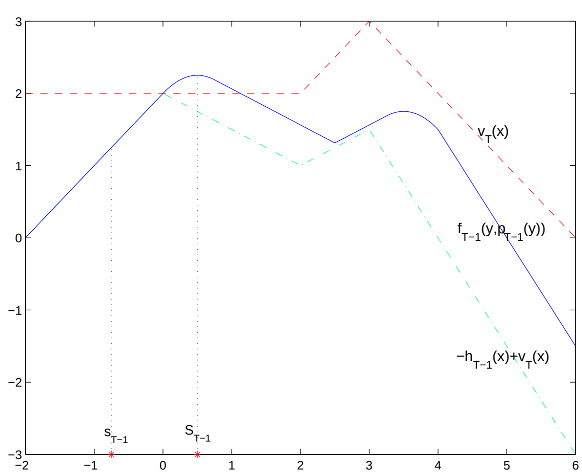

Figure 1: $v_T(y)$,$f_{T-1}(y, p_{T-1}(y))$ and $v_T(y) - h_{T-1}(y)$

Figure 1 depicts the functions $v_T(y)$, $v_T(y) - h_{T-1}(y)$ and $f_{T-1}(y, p_{T-1}(y))$ while Figure 2 presents the optimal selling price $p_{T-1}(y)$. In Figure 2, the dash-dotted line is $p_{T-1}(y)$

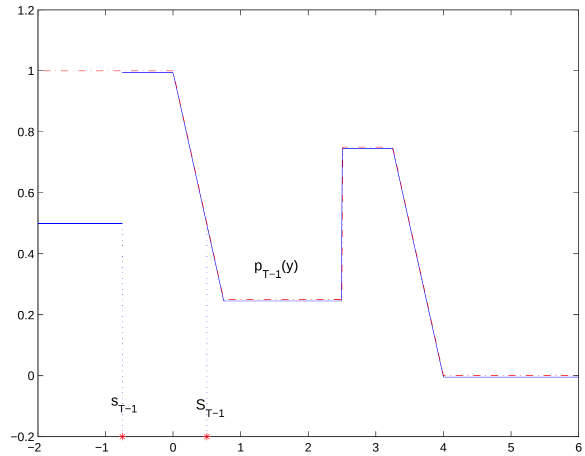

Figure 2: $p_{T-1}(y)$

before making the decision to order up to S_{T-1} and solid line represents the optimal price after making the ordering decision.

For instance, if
$y=1,\ y-1+p=p\in[0,1],\ f_{T-1}(1,p)=\frac{1}{2}p-p^2+2$ and $p_{T-1}(1)=\frac{1}{4}$,
while when
$y=3,\ y-1+p=2+p\in[2,3],\ f_{T-1}(3,p)=\frac{3}{2}p-p^2+1$ and $p_{T-1}(3)=\frac{3}{4}$.

8 Appendix B

Proof of Lemma 3: Consider an instance with stationary input data for the last two periods of problem (3): for $t=T,T-1$,

$$\begin{array}{l} c_t=0,\beta_t=0,\underline{d}_t=0,\bar{d}_t=b,D_t(p)=b-ap,\\ R_t(d)=d(b-d)/a,h_t(x)=h_+\max\{0,x\}+h_-\max\{0,-x\}, \end{array}$$

and

$$\alpha_t=\left\{\begin{array}{ll}\underline{\alpha}, & \text{with probability } q\\ \bar{\alpha}, & \text{with probability } 1-q,\end{array}\right.$$

where $h_+>>h_->0$ are fixed, $\bar{\alpha}>1>\underline{\alpha}>0$ and $q\underline{\alpha}+(1-q)\bar{\alpha}=1$. We will choose $b>>h_+$ and $a>>b^2$.
<u>For period T</u>: Given $h_+>>h_->0$, choose $b>>h_+$ and $a>>b^2$. In this case, it is optimal to choose a feasible d such that $y-\underline{\alpha}d$ is as close to 0 as possible. Therefore,

$$d_T(y)=\left\{\begin{array}{ll}0, & \text{for } y\le 0,\\ y/\underline{\alpha}, & \text{for } 0\le y\le\underline{\alpha}b,\\ b, & \text{for } y\ge\underline{\alpha}b,\end{array}\right.$$

and

$$g_T(y, d_T(y)) = \begin{cases} h_- y, \text{ for } y \le 0, \\ y/\underline{\alpha}(b - y/\underline{\alpha})/a + h_-(y - y/\underline{\alpha}), \text{ for } 0 \le y \le \underline{\alpha} b, \\ -q h_+(y - \underline{\alpha} b) + (1-q) h_-(y - \bar{\alpha} b), \text{ for } \underline{\alpha} b \le y \le \bar{\alpha} b, \\ -h_+(y - b), \text{ for } y \ge \bar{\alpha} b. \end{cases}$$

Under these assumptions, we have that $S_T = 0$, $g_T(0,0) = 0$ and

$$v_T(x) = \begin{cases} -k, \text{ for } y \le -k/h_-, \\ g_T(y, d_T(y)), \text{ for } y \ge -k/h_-. \end{cases}$$

<u>For period $T-1$:</u> We have that

$$-h_{T-1}(x) + v_T(x) = \begin{cases} -k + h_- x, \text{ for } x \le -k/h_-, \\ 2h_- x, \text{ for } -k/h_- \le x \le 0, \\ x/\underline{\alpha}(b - x/\underline{\alpha})/a + h_-(x - x/\underline{\alpha}) - h_+ x, \text{ for } 0 \le x \le \underline{\alpha} b, \\ -q h_+(x - \underline{\alpha} b) + (1-q) h_-(x - \bar{\alpha} b) - h_+ x, \text{ for } \underline{\alpha} b \le x \le \bar{\alpha} b, \\ -2h_+ x + h_+ b, \text{ for } x \ge \bar{\alpha} b. \end{cases}$$

Since $b >> h_+ >> h_- > 0$ and $a >> b^2$, it is optimal to choose a feasible d such that $y - \underline{\alpha} d$ is as close to 0 as possible. Therefore,

$$d_{T-1}(y) = d_T(y),$$

and

$$g_{T-1}(y, d_{T-1}(y)) = \begin{cases} -k + h_- y, \text{ for } y \le -k/h_-, \\ 2h_- y, \text{ for } -k/h_- \le y \le 0, \\ y/\underline{\alpha}(b - y/\underline{\alpha})/a + 2h_-(y - y/\underline{\alpha}), \text{ for } 0 \le y \le \underline{\alpha} k/(h_-(\bar{\alpha} - \underline{\alpha})), \\ y/\underline{\alpha}(b - y/\underline{\alpha})/a + h_-(y - y/\underline{\alpha}) - (1-q)k, \text{ for } \underline{\alpha} k/(h_-(\bar{\alpha} - \underline{\alpha})) \le y \le \underline{\alpha} b, \\ \cdots, \text{ for } y \ge \underline{\alpha} b. \end{cases}$$

Observe that $g_{T-1}(y, d_{T-1}(y))$ is decreasing for $y \ge 0$ and directionally differentiable for any y. For $y \ge \underline{\alpha} b$, the directional differential of $g_{T-1}(y, d_{T-1}(y))$ is much larger than that for $y < \underline{\alpha} b$, since $h_+ >> h_- > 0$.

Denote by

$$\underline{y} = \underline{\alpha} k/(h_-(\bar{\alpha} - \underline{\alpha})) = \lambda \underline{\alpha} b,$$

for some $\lambda \in [0,1]$. For $\underline{y} \le y \le \underline{\alpha} b$, we have that

$$g_{T-1}(y, d_{T-1}(y)) = y/\underline{\alpha}(b - y/\underline{\alpha})/a + h_-(y - y/\underline{\alpha}) - (1-q)k.$$

It remains to show that $g_{T-1}(y, d_{T-1}(y))$ is not k-concave. Observe that for $y = 0$, $d_{T-1}(y) = 0$, and $g_{T-1}(0,0) = 0$. If $g_{T-1}(y, d_{T-1}(y))$ is k-concave, then for $x_0 = 0$, $x_1 = \underline{\alpha} b$, we have from the definition of k-concavity that

$$\underline{y}/\underline{\alpha}(b - \underline{y}/\underline{\alpha})/a + h_-(\underline{y} - \underline{y}/\underline{\alpha}) - (1-q)k \ge \lambda(h_- b(\underline{\alpha} - 1) - (1-q)k) - \lambda k,$$

which implies that

$$\underline{y}/\underline{\alpha}(b - \underline{y}/\underline{\alpha})/a - (1-q)k \ge -\lambda(2-q)k.$$

However, if we increase a, b and keep b^2/a very small, the above inequality does not hold since $\lambda \to 0+$, which is a contradiction. Hence $g_{T-1}(y, d_{T-1}(y))$ is not k-concave. Furthermore, under the above assumptions, one can see that $S_{T-1} = 0$ and $g_{T-1}(0, 0) = 0$. Therefore $v_{T-1}(x)$ is not k-concave, since $v_{T-1}(x) = g_{T-1}(x, d_{T-1}(x))$ for $x \geq 0$. ■

9 Appendix C

Proof of Lemma 4: We extend the example of Appendix B by investigating time period $T-2$. Note that

$$v_{T-1}(x) = \begin{cases} -k, \text{ for } \; y \leq -k/(2h_-), \\ g_{T-1}(y, d_{T-1}(y)), \text{ for } \; y \geq -k/(2h_-). \end{cases}$$

<u>For period $T-2$:</u> Let

$$c_{T-2} = \beta_{T-2} = 0, \underline{d}_{T-2} = 0, \bar{d}_{T-2} = b', D_{T-2}(p) = b' - a'p, R_{T-2}(p) = d(b'-d)/a',$$

and

$$h_{T-2}(x) = \rho \max\{0, -x\} + \epsilon \max\{0, x\}.$$

We choose appropriate ρ, ϵ, a' and b' such that $\rho >> a', b', h_+$, and ϵ is sufficiently small. Under these assumptions, it is optimal to choose a feasible d such that $y - \bar{\alpha} d$ is as close to 0 as possible and keep $y - \bar{\alpha} d$ nonnegative since $-h_{T-2}(x) + v_{T-1}(x)$ is non-increasing for $x \geq 0$. Specifically,

$$d_{T-2}(y) = y/\bar{\alpha}, \quad \text{for } \; 0 \leq y \leq \underline{\alpha} b.$$

Denote by

$$y^* = \bar{\alpha}(b' - (\gamma + \epsilon')a')/2,$$

and

$$\hat{y} = \bar{\alpha}(b' - (2\gamma + \epsilon')a')/2,$$

where $\gamma = q(\bar{\alpha} - \underline{\alpha})h_-(1/\underline{\alpha} - 1)$ and $\epsilon' = q(\bar{\alpha} - \underline{\alpha})\epsilon$.

In order to simplify notation, we omit the term $y/\underline{\alpha}(b - y/\underline{\alpha})/a$ in $g_{T-1}(y, d_{T-1}(y))$ for $0 \leq y \leq \underline{\alpha} b$. This is possible, since $b^2 << a$ implying that $y/\underline{\alpha}(b - y/\underline{\alpha})/a \to 0_+$ and thus does not impact the argument below.

If

$$\underline{y} \leq (1 - \underline{\alpha}/\bar{\alpha})y \leq \underline{\alpha} b, \tag{15}$$

then

$$g_{T-2}(y, d_{T-2}(y)) = y/\bar{\alpha}(b' - y/\bar{\alpha})/a' - (\gamma + \epsilon')y/\bar{\alpha} - q(1-q)k.$$

So if

$$\underline{y} \leq (1 - \underline{\alpha}/\bar{\alpha})y^* \leq \underline{\alpha} b, \tag{16}$$

one can see from the first order optimality condition that y^* maximizes $g_{T-2}(y, d_{T-2}(y))$ for y satisfying (15) and

$$g_{T-2}(y^*, d_{T-2}(y^*)) = (b' - (\gamma + \epsilon')a')^2/(4a') - q(1-q)k.$$

For

$$0 \le y \le \underline{y}/(1-\underline{\alpha}/\bar{\alpha}), \tag{17}$$

we have that

$$g_{T-2}(y, d_{T-2}(y)) = y/\bar{\alpha}(b' - y/\bar{\alpha})/a' - (2\gamma + \epsilon')y/\bar{\alpha},$$

and if

$$\hat{y} < 0, \tag{18}$$

then $y = 0$ maximizes $g_{T-2}(y, d_{T-2}(y))$ for y satisfying (17), since $g'_{T-2}(\hat{y}, d_{T-2}(\hat{y})) = 0$. If y^* satisfies (16) and

$$g_{T-2}(y^*, d_{T-2}(y^*)) = (1-\delta)k \tag{19}$$

for some $\delta \in (0,1)$, then y^* is the global maximizer of $g_{T-2}(y, d_{T-2}(y))$ since $\rho >> h_+ >> h_-$. Finally, if in addition to (16), (18) and (19), we have

$$g_{T-2}(\underline{y}, d_{T-2}(\underline{y})) < -\delta k, \tag{20}$$

then we know that it is optimal to order up to y^* when the inventory level is $\underline{y}$ and not to order when the inventory level is $y = 0$, since $g_{T-2}(0, d_{T-2}(0)) = g_{T-2}(0,0) = 0$. This implies that

$$s_{T-2} < 0 < \underline{y} < S_{T-2} = y^*,$$

and therefore, any (s, S) inventory policy is not optimal in this case.

The remaining task is to check whether (16), (18), (19) and (20) can hold simultaneously by choosing the appropriate parameters. Note that (19) is equivalent to

$$b' = (\gamma + \epsilon')a' + 2\sqrt{(1 + p(1-p) - \delta)ka'}, \tag{21}$$

and the above equation, together with inequality (18) and the definition of $\hat{y}$, gives that

$$4(1 + q(1-q) - \delta)k/\gamma^2 < a'. \tag{22}$$

By the definition of $\underline{y}$ and (21), (20) is equivalent to

$$\gamma - (2\sqrt{(1 + p(1-p) - \delta)ka'} - \underline{y}/\bar{\alpha})/a' > h_-(\bar{\alpha} - \underline{\alpha})\bar{\alpha}/\underline{\alpha}\delta. \tag{23}$$

Now it is clear that there exists δ sufficiently small, a', b' sufficiently large compared with $k, \underline{y}, h_-$ and $a', b' << \rho, b$ such that (16), (21), (22) and (23) hold.

Therefore, the example shows that (s, S, p) policies are not necessarily optimal. ■

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