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# Effective Dynamic Pricing Strategies with Stochastic Demand

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July 2001

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# Effective Dynamic Pricing Strategies with Stochastic Demand\*

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## Abstract

We consider jointly determining prices and production values in a multiple period horizon under a general, non-stationary stochastic demand function. Additional key assumptions are that the available production capacity is limited, unmet demand is lost, and sales are discretionary. We analyze and compare partial planning and delayed strategies, in which a decision may be delayed until demand is realized in previous periods. For example, in Delayed Production (Delayed Pricing), pricing (production) is determined at the beginning of the horizon, and the production (pricing) decision is made at the beginning of each period before customer orders are received. A special case of our model is where a single price is chosen over the horizon. We develop heuristics for the strategies based on deterministic approximations and analyze the worst-case performance. Computational analysis is performed to develop insights about the performance of the heuristics.

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## 1 Introduction

In recent years, a number of industries have used innovative pricing strategies to manage their inventory effectively. For example, techniques such as *revenue management* have been applied in service industries as varied as the airlines, hotels, and rental car agencies—integrating price, inventory control, and quality of service. In other cases such as the power industry, varying price according to demand or time of day is becoming a tool to utilize capacity more efficiently. The retail industry has also concentrated on coordinating pricing and inventory control, particularly focusing on using price as a market clearing mechanism.

However, the integration of pricing with production in a manufacturing setting is still in its early stages. There are a number of characteristics that distinguish general manufacturing industries from the industries mentioned previously, including the non-perishability of products and the ability to vary production levels. Furthermore, manufacturing differs from most retail environments in its reordering and capacity characteristics. Manufacturing generally has production in every period that is limited by the capacity of the system, while retailing often involves a single large order at the start of a selling season.

Our objective in this research is thus to analyze models and problems where pricing decisions are considered jointly with production decisions, incorporating the relationship between price and demand, and taking into account capacity limitations. In particular, we are interested in *partial planning* strategies in a multi-period setting under stochastic demand, where products are non-perishable or have a long selling season. In partial planning strategies, one decision (price or production) is made at the beginning of horizon, and the second decision (production or price) is made at the beginning of each period—after the realization of demand uncertainty in previous periods but before demand is realized in the current period. We consider discretionary sales, where inventory may be saved to satisfy future demand. The strategies that we analyze are driven by industrial examples such as the following.

1. A supplier of manufactured parts has variability in raw material supply and uses prices to better match demand and supply in each period. The supplier contracts with a manufacturer over a time horizon, offering the manufacturer fixed prices in advance for planning purposes, but allowing orders to be placed in each period due to the manufacturer's high inventory holding cost and unpredictable demand. The supplier adjusts production in each time period based on previous inventory and expected orders.
2. A manufacturer needs to determine procurement decisions in advance in order to sign contracts with suppliers for part delivery. Thus, the manufacturer determines a production schedule at the beginning of a time horizon, but makes decisions regarding available inventory and price on a period by period basis.
3. A manufacturer whose primary distribution channel is through catalogs determines a single constant price over the lifetime of the product and wants the profit maximiz-

ing price. Production decisions are determined period by period, based on expected demand in present and future periods as well as inventory from previous periods.

These examples demonstrate the strategies that we are interested in, namely ones combining pricing and production decisions under uncertain demand. We focus our efforts on *dynamic pricing* models, where price changes over time in response to changes in supply or demand, but we also address *fixed pricing* models in some cases, where price is constant over time. We are particularly interested in partial planning models, where one or more decisions must be made at the beginning of a horizon.

In example one, the decision making firm determines prices for a planning horizon a priori and determines the production decision based on the state of the system and forecast demand. The firm is able to vary the production level based on inventory left over from previous periods or produce to satisfy demand in future periods. We refer to this strategy as *Delayed Production*.

Example two illustrates another partial planning model. The manufacturer plans production at the beginning of the horizon but makes the price decision on a period by period basis. We refer to this strategy as *Delayed Pricing*. In this case price can be used partially as a market clearing mechanism to deal with inventory from previous periods, basically the inventory that results from uncertain demand.

In some cases, a firm may believe that selecting a constant price for a product is the best strategy. However, procurement flexibility in each period may still be desired. This is illustrated in the last example, and the strategy is referred to as *Fixed Pricing*. As before, the price chosen should account for demand uncertainty over the horizon as well as demand seasonality (or non-stationarity) over time. This pricing strategy is often applied when goods are non-perishable or have a long selling season, e.g. furniture products.

In all of the examples described above, it may be profitable to apply *discretionary sales*, that is, to set aside inventory to satisfy future demand, even if the decision means losing sales in the current period. Although choosing to lose sales may seem counter to making profit, the inventory is set aside in situations when it is likely to generate a larger income in the future. This would typically occur if the price in the future is higher or if the future production costs were high.

The following assumptions are common to all the models considered in the paper. We assume that demand is non-stationary over time and that the relationship between price and expected demand is known. Demand uncertainty in each period may vary with price or be price-independent; there are no other assumptions on the relationship between price and demand or on the distribution of the random component of the demand function. Production is limited by the available capacity in each period, which is known and may vary over time. If product is not available to meet customer demand, we assume the sale is lost; inventory may be saved for the future even if this results in lost sales. We focus on the pricing and production decisions and the impact of dynamic pricing on the supply chain under the assumption of a monopolistic decision-making firm. The goal is to maximize the expected profit, which comprises revenue, inventory holding cost and production cost, over a finite horizon.

The paper is organized as follows. We review the related literature in Section 1.1 and describe the notation and problem assumptions in Section 2.1. In Section 2.2 we analyze Delayed Production. We show that given a price vector, the optimal production and inventory policy has a special structure that allows us to efficiently find the optimal policy.

Specifically, we show that the optimal policy is characterized by two parameters, an order-up-to level and a save-up-to level, both of which are modified base-stock policies. For the order-up-to-level, the policy is to produce such that the inventory level reaches the order-up-to amount if there is sufficient capacity, otherwise produce to maximum capacity. The corresponding policy for save-up-to is to set aside that amount of inventory for the future if it is available; otherwise, set aside as much inventory as possible for the future. Both of these policies are time-dependent but are independent of the inventory level at the beginning of each period and of the realized customer demand; thus, the decision to fulfill orders may be made as customers arrive. Our results also imply that a simple search on all possible prices finds the optimal Fixed Price policy. Finally, we suggest an heuristic to generate an effective pricing vector for the general Delayed Production Strategy.

In Section 2.3, we analyze Delayed Pricing, in particular studying the relationship between optimal price and inventory level at the beginning of each period. We show that the optimal period price is not a decreasing function of inventory, thus making the Delayed Pricing problem a difficult problem. Additionally, we show that determining the available sales before demand is realized is inferior to determining it after demand is realized. Therefore, given a production schedule, we provide a dynamic program that generates the optimal pricing vector, and we suggest an heuristic for determining a production schedule.

In Section 3, we perform computational analysis using data from a manufacturing partner. We are particularly interested in the comparison between the Delayed Production and Delayed Pricing strategies; the analysis shows that Delayed Pricing is usually more profitable. Exceptions to this are when production cost is high and the available capacity is high or under certain types of demand seasonality. We also compare the two partial planning strategies to the Fixed Price strategy and show that Dynamic Pricing almost always results in higher expected profit. The computational analysis for Partial Planning is particularly useful in estimating the magnitude of profit increase from dynamic pricing; our results indicate the profit increase may be as much as 2 - 7 % compared to fixed pricing.

In the final section, Section 4 we describe some extensions to the stochastic pricing problem and make conclusions. We also suggest future directions for research in this area.

## 1.1 Literature Review

To prepare the background for the pricing problem analyzed in this paper, we must consider research in a variety of areas including inventory control, marketing, revenue management research that pertains to pricing models.

### 1.1.1 Pricing and Inventory Control

Joint pricing and inventory control strategies in a manufacturing environment were first considered by Whitin (1955). In this paper, Whitin examined a single period problem, most similar to a "newsboy" problem, and determines a single price and supply quantity. Other researchers such as Mills (1959), Karlin and Carr (1962), Hempenius (1970), Lau and Lau (1988) and Polatoglu (1991) have considered similar single period problems under various demand conditions. Petruzzi and Dada (1999) review and extend results on the newsvendor problem with pricing. They provide a unified framework to compare the price under a deterministic setting with the price under uncertainty. A main result demonstrated in that paper (as well as earlier papers such as Mills (1959)), indicates that under additive demand the price under uncertainty is lower than the "riskless" price (under the assumption of deterministic demand).

Price determination and restocking in a multi-period setting has also been considered by a number of researchers. For example, Thowsen (1975), and Zabel (1972) both consider multi-period models with a convex order cost structure. The more difficult case of concave costs has been examined by Kunreuther and Richard (1971), Thomas (1970), Wagner (1960) and Wagner and Whitin (1958) among others. Although most of the multi-period models with concave costs assume demand to be deterministic, Thomas (1974) develops a version of the Wagner-Whitin (1958) lot-sizing algorithm under conditions of stochastic demand. A thorough review of both single and multi-period models combining pricing and inventory strategies can be found in Eliashberg and Steinberg (1991). Rajan et al. (1992) study a multi-period problem with decaying inventory, and Abad (1996) extends these results to allow for backlogging. In more recent papers, Bhattacharjee and Ramesh (2000) adapt the Wagner-Whitin method to solve a multi-period pricing problem; they present efficient search heuristics and study structural properties of the model. Polatoglu and Sahin (2000) provide sufficient conditions for an  $(s, S)$  policy to be optimal for a price and procurement problem.

Some research on pricing and inventory control of product, particularly in the retail industry, has been done under the name of revenue management. For instance, Gallego and van Ryzin (1994) analyze the dynamic adjustment of price as a function of inventory and length of remaining sales period; the demand is stochastic but no restocking is allowed. In a similar problem, Gallego and van Ryzin (1997) apply revenue management to a network and determine bounds on the optimal solution by analyzing the deterministic case. Bitran and Mondschein (1997) also consider price as a function of inventory and time; they compare the structure of the optimal policies to those observed in practice and discuss various economic insights. Subrahmanyam and Shoemaker (1996) developed a model that incorporates learning of demand while determining optimal pricing and stocking policies.

As far as we can tell, most research on integrating pricing strategies with inventory control policies have ignored production capacity limitations. One exception to this is Lai (1990), who considers the issue of whether capacity constraints contribute to asymmetry in price behavior. The most notable exception, however, is the work by Federgruen and Heching (1999), who address the problem of determining optimal pricing and inventory control strategies. Their research indicates a method for determining an optimal order up

to policy. One property they show is that price decreases with increasing initial inventory. Their model is somewhat similar to our model; however, they assume backlogging whereas our model assumes lost sales. Another important distinction is that they assume “demand in each period is concave in the period’s price” while we assume a general function. Additionally, they are focused on making decisions in each period, whereas we consider models with a planning element. Finally, in our model we show that price does not necessarily decrease with increasing initial inventory.

In addition to papers combining pricing and inventory, it is important to consider traditional inventory models; these are particularly relevant to our Delayed Production strategy. Zipkin (2000) provides an excellent review of literature on inventory control. A multi-period inventory model with non-stationary demand, varying production cost and lost sales is most closely related to the stochastic pricing problems considered in this paper (see Chapter 9 in Zipkin). Of course, in this model, as well as the traditional inventory literature, demand is always satisfied when inventory exists, while in our model it is possible to decide not to satisfy demand even when inventory is available. This difference between our work and the traditional inventory literature leads to the concept of save-up-to policy which is an important strategy in our partial planning models.

Recent work by Scarf (2000) addresses an inventory model when sales are discretionary, as we also assume. Scarf considers a model with production capacity limits and set-up costs where price is given and inventory policy is the decision factor; this is similar to our Delayed Production strategy after a pricing decision has been made. The optimal policy for his model is of the  $(s,S)$  form, and the optimal discretionary sales is dependent on realized demand, whereas in our Delayed Production Strategy the optimal policy has an order-up-to and save-up-to targets in each period, which are both independent of initial inventory and realized demand.

### 1.1.2 Specific Pricing Strategies

Many of the pricing and inventory papers in a multi-period setting are focused on finding a price in each period of the horizon, however some research has also been directed towards finding a single optimal price over an multi-period horizon. Kunreuther and Schrage (1973) provide upper and lower bounds on the optimal price, whereas Gilbert (1999) finds the optimal price under a less general model of demand. Gilbert (2000) considers a problem with multiple products sharing common production capacity and demonstrates a procedure for finding the optimal fixed price for each product. In the first two papers, production has a set-up cost, whereas in the multiple product work there is no set-up cost. Little work has been done on the fixed pricing problem with stochastic demand. This research is pertinent to the Fixed Pricing strategy we discuss.

Van Mieghem and Dada (1999) is another important paper relevant to our work, in which the authors explicitly consider price postponement versus production postponement strategies. They focus on a single-period, two-stage process with an initial decision, e.g. production decision, followed by a realization of demand, followed by another decision, e.g.

pricing decision. Thus, price (production) postponement as outlined by Van Mieghem and Dada is different from Delayed Pricing (Production) in our model since in their case the postponed decisions are made after demand is realized. They find that conditions dictate whether price postponement or production postponement is more valuable to a firm. Specifically they show that the former is likely to be more valuable if demand variability, marginal production, and holding costs are low. Their paper also addresses the decision of capacity investment and considers competition.

## 2 The Stochastic Pricing Problems

We analyze a multi-period single manufacturer model for a single product, where pricing and production decisions must be made for each period. We focus on partial planning strategies, where one decision (e.g., production) is made at the beginning of the horizon for all periods, and where a second decision (e.g., pricing) is made in each period based on the state of the system. We focus on situations where there may be demand seasonality over time represented by time-dependent expected demand curves as well as an additional stochastic element that is a random component with a known distribution.

### 2.1 Notation and Assumptions

We make a number of assumptions common to all planning models considered. Periods are indexed consecutively from  $1, 2, \dots, T$ . The production quantity in period  $t$  is limited by  $q_t$  for  $t = 1, 2, \dots, T$ . The cost of producing a unit is  $k_t$ , and inventory holding cost,  $h_t$ , is charged to carry inventory from period  $t - 1$  to period  $t$ . The salvage value at the end of the horizon is represented as  $v$ , which we assume is less than the price charged in the last period.

In both the Delayed Production and Delayed Pricing Strategies, we assume that demand in each period is a non-stationary, general stochastic function,  $d_t(P_t, \epsilon_t(P_t))$ , that depends on the price in period  $t$ ,  $P_t$ , and a random term with a known distribution that may also depend on price,  $\epsilon_t(P_t)$ . Furthermore, we assume demand in each period is independent and that expected demand in period  $t$  for a price  $P_t$ ,  $\bar{d}_t(P_t)$ , is decreasing in  $P_t$ . We do not assume a particular distribution on demand. Price in period  $t$  is bounded below and above by  $p_t^{min}$  and  $p_t^{max}$  respectively, implying corresponding bounds on the expected demand in period  $t$ ,  $\bar{d}_t^{max}$  and  $\bar{d}_t^{min}$ .

In each period, we need to decide on price,  $P_t$ , and production quantity,  $X_t$ . Let  $I_t$  represent the inventory available at the beginning of period  $t$ , and  $D_t$  is the satisfied demand in each period. Let the unindexed variables  $P$ ,  $X$ ,  $I$ , and  $D$  represent the vectors of price, production, inventory and satisfied demand, respectively, over the entire horizon.

As in classic inventory literature, we define  $Y_t$  as the inventory level after production and before demand is realized. Finally, and unlike in the traditional inventory literature, we allow for discretionary sales, where the facility does not have to satisfy a unit of demand even if inventory is available. Specifically, we assume that at the beginning of every period,

the facility can decide on the amount of inventory  $S_t$  to save for future sales. (In fact, we will also analyze the impact of allowing the decision maker to decide how many customers to reject after observing all or a portion of the period's demand.)

Thus, in each period, the inventory available for customers is determined by three parameters:

1. The initial inventory at the beginning of the period;
2. Production level in this period; and
3. The amount of inventory to be saved for future periods.

Thus, the inventory available for customers in the current period is determined before the actual demand,  $d_t$ , is observed and hence for every period  $t$ ,  $D_t \leq d_t$ .

We use the above notation to introduce various partial planning strategies.

## 2.2 Delayed Production Strategy

In the delayed production strategy, the firm determines period dependent prices at the beginning of the horizon. The firm determines production on a period by period basis, incorporating demand realizations in previous periods but not in the current period. For a given price vector  $P$ , the problem of determining production and sales is defined as a Markov Decision problem with the initial inventory ( $I$ ) in a period as the state of the system.

Let  $J_t(I)$  represent the expected profit from the beginning of period  $t$  until the end of the time horizon. We use the phrase "profit-to-go" to refer to profit in the current and future periods. Let  $G_t(Y)$  represent the expected profit-to-go with  $Y$  units of product available. That is,  $G_t(Y)$  includes revenue and inventory holding costs for the current period but does not include production cost;  $G_t(Y)$  also includes the expected profit-to-go for future periods,  $J_{t+1}$ . We denote  $\Delta J_t(I)$  and  $\Delta G_t(I)$  as the marginal change in respective profits when inventory increases from  $I - 1$  to  $I$  units. Let the cumulative demand distribution for a given price  $P_t$  in period  $t$  be  $\Psi_t^{P_t}$ , and let the corresponding demand distribution be  $d\Psi_t^{P_t}$  or  $\psi_t^{P_t}$ .

Given a price vector, the optimal expected profit in period  $t$  is as follows:

$$J_t(I) = \max_{Y_t: I_t \leq Y_t \leq I_t + q_t} -k_t(Y_t - I_t) + G_t(Y_t), \quad (1)$$

where

$$G_t(Y_t) = \max_{S_t: 0 \leq S_t \leq Y_t} \left\{ \int P_t(\min(D_t, Y_t - S_t)) d\Psi_t^{P_t}(D_t) - h_{t+1}S_t - \int h_{t+1}(\max(0, Y_t - S_t - D_t)) d\Psi_t^{P_t}(D_t) + \int J_{t+1}(S_t + \max(0, Y_t - S_t - D_t)) d\Psi_t^{P_t}(D_t) \right\}. \quad (2)$$

In the case where  $t$  is the last period in the horizon, the last term in equation (2) is replaced with the expected salvage value of leftover inventory given by  $\int v(\max(0, D_T - (Y_T - S_T))) d\Psi_T^{P_T}(D_T)$ .

The first term in (1) is production cost, and the first term in (2) is the revenue, where sales is the minimum between realized demand and the available inventory after production in period  $t$ . The next two terms in (2) represent the inventory holding cost. The first of these is the holding cost for the inventory that has been set aside for the future, and the latter of the two accounts for holding cost when realized demand is less than the available inventory,  $Y_t - S_t$ . Finally, the last term in the second equation represents profit-to-go, given an inventory level at the beginning of period  $t + 1$  equal to the inventory set aside as well as inventory that was unsold due to low demand.

In the next section, we show that this problem has structural properties that can be used to obtain a solution efficiently. Specifically, we show that the expected profit in period  $t$ , for  $t = 1, 2, \dots, T$  is a concave function of inventory ( $I_t$ ). This property is useful in showing the existence of the optimal order-up-to and save-up-to inventory levels, which we will denote by  $Y_t^*$  and  $S_t^*$ , respectively.

### 2.2.1 Analysis of Delayed Production for a Given Price Vector

Suppose that a vector of prices  $P$  is determined at the beginning of the horizon. The Delayed Production Strategy then becomes a decision of scheduling production ( $X_t$ ) and determining the inventory to set aside for the future,  $S_t$ , in each period in a way that maximizes profit. We shall first address the Delayed Production Strategy assuming a given price vector, then discuss finding a price vector.

We prove the following results for the Delayed Production Strategy:

#### Theorem 2.1 :

- For all  $t = 1, \dots, T$ ,  $\Delta G_t(Y)$  is non-increasing in  $Y$ , and thus  $G_t(Y)$  is concave;
- For all  $t = 1, \dots, T$ ,  $J_t(I)$  is concave in  $I$ .

*Proof.*

1. We see from equation (2) that  $\Delta G_T(Y) = P_T[1 - \Psi_T^{Pr}(Y - 1)] + v\Psi_T^{Pr}(Y - 1)$ . Thus it is clear that  $\Delta G_T$  is non-increasing since,
 
$$\Delta G_T(Y) - \Delta G_T(Y + 1) = P_T\psi_T^{Pr}(Y) - v\psi_T^{Pr}(Y) > 0$$
2. Given  $t$ ,  $t = 1, \dots, T$ , assume that  $\Delta G_t$  is non-increasing. We show that  $J_t$  is concave.
  - Define  $Y_t^*$  as 0 if  $k_t > \Delta G_t(1)$  else as  $\max \{Y : Y > 0 \text{ and } k_t \leq \Delta G_t(Y)\}$ .
  - Hence,

$$J_t(I) = \begin{cases} -k_t q_t + G_t(I + q_t) & \text{if } I \leq Y_t^* - q_t \\ -k_t(Y_t^* - I) + G_t(Y_t^*) & \text{if } Y_t^* - q_t < I \leq Y_t^* \\ G_t(I) & \text{if } Y_t^* < I \end{cases}$$

- Since  $\Delta G_t$  is non-increasing, and thus  $G_t$  is concave, and by the choice of  $Y_t^*$ ,  $\max(0, \min(q_t, Y_t^* - I_t))$  is the  $Y_t$  that maximizes  $J_t(I_t)$ .
- We consider these cases and show that  $\Delta J_t$  is non-increasing within each of them:
  - (a) Case 1:  $I \leq Y_t^* - q_t$ :  
 $J_t(I) - J_t(I - 1) = G_t(I + q_t) - G_t(I + q_t - 1) \geq k_t$  because of the choice of  $Y_t^*$ . Since  $\Delta G_t$  is non-increasing, so is  $\Delta J_t(I)$  for this case.
  - (b) Case 2:  $Y_t^* - q_t < I \leq Y_t^*$ :  
 $J_t(I) - J_t(I - 1) = k_t$ .  $\Delta J_t(I)$  is constant, and thus non-increasing in this case as well.
  - (c) Case 3:  $I > Y_t^*$ :  
 $J_t(I) - J_t(I - 1) = G_t(I) - G_t(I - 1) \leq k_t$  because of the choice of  $Y_t^*$ . Again, since  $\Delta G_t$  is non-increasing, so  $\Delta J_t(I)$  is non-increasing.
- To see that  $\Delta J_t$  is non-increasing for all  $I$ , we compare the three cases as  $I$  increases. We have that  $\Delta J_t$  is biggest when  $I$  is small ( $\geq k_t$ ), it decreases to  $k_t$  in the second case, and when  $I$  is largest,  $\Delta J_t$  is smallest ( $\leq k_t$ ). Thus  $\Delta J_t$  is non-increasing, and  $J_t$  is concave.

3. Given that  $\Delta G_t$  is non-increasing and  $J_t$  is concave for  $t = t, \dots, T$ , we now show that  $\Delta G_{t-1}$  is non-increasing.

- Define  $S_{t-1}^*$  as 0 if  $P_{t-1} > \Delta J_t(1) - h_t$  else as  $\max\{I : I > 0 \text{ and } P_{t-1} \leq \Delta J_t(I) - h_t\}$ .
- Hence,

$$\Delta G_{t-1}(Y) = \begin{cases} \Delta J_t(Y) - h_t & \text{if } Y \leq S_{t-1}^* \\ P_{t-1} \left[ 1 - \Psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^* - 1) \right] & \text{if } Y > S_{t-1}^* \\ \quad + \sum_{k < Y - S_{t-1}^*} [\Delta J_t(Y - k) - h_t] \psi_{t-1}^{P_{t-1}}(k). & \end{cases}$$

- Since  $J_t$  is concave and  $\Delta J_t$  is non-increasing,  $\max(S_{t-1}^*, Y_{t-1})$  is the  $S$  that maximizes  $G_{t-1}(Y_{t-1})$ .
- We consider  $\Delta G_{t-1}$  within several cases:

(a) Case 1:  $Y < S_{t-1}^*$ :  
 $\Delta G_{t-1}(Y) - \Delta G_{t-1}(Y + 1) = \Delta J_t(Y) - \Delta J_t(Y + 1) \geq 0$  due to the concavity of  $J_t$ .

(b) Case 2:  $Y > S_{t-1}^*$ :  
 $\Delta G_{t-1}(Y) - \Delta G_{t-1}(Y + 1) =$   
 $P_{t-1} \psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) - \left[ \Delta J_t(S_{t-1}^* + 1) - h_t \right] \psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*)$   
 $+ \sum_{k < Y - S_{t-1}^*} [\Delta J_t(Y - k) - \Delta J_t(Y - k + 1)] \psi_{t-1}^{P_{t-1}}(k) \geq 0.$

This is true since

- $P_{t-1} \psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) - \left[ \Delta J_t(S_{t-1}^* + 1) - h_t \right] \psi_{t-1}^{P_{t-1}}(Y - S_{t-1}^*) \geq 0$  because of the choice of  $S_{t-1}^*$ , and

ii.  $\sum_{k < Y - S_{t-1}^*} [\Delta J_t(Y - k) - \Delta J_t(Y - k + 1)] \psi_{t-1}^{P_{t-1}}(k) \geq 0$  for  $Y > S_{t-1}^* \geq 0$   
 due to the concavity of  $J_t$ .

(c) Case 3:  $Y = S_{t-1}^*$ :  
 $\Delta G_{t-1}(S_{t-1}^*) - \Delta G_{t-1}(S_{t-1}^* + 1)$   
 $= \Delta J_t(S_{t-1}^*) - h_t - P_{t-1} [1 - \Psi_{t-1}^{P_{t-1}}(0)] - [\Delta J_t(S_{t-1}^* + 1) - h_t] \psi_{t-1}^{P_{t-1}}(0)$   
 $= [\Delta J_t(S_{t-1}^*) - h_t - P_{t-1}] [1 - \psi_{t-1}^{P_{t-1}}(0)]$   
 $+ [\Delta J_t(S_{t-1}^*) - \Delta J_t(S_{t-1}^* + 1)] \psi_{t-1}^{P_{t-1}}(0) \geq 0$  due to the choice of  $S_{t-1}^*$  and the  
 concavity of  $J_t$ .

- Since  $\Delta G_{t-1}(Y) - \Delta G_{t-1}(Y + 1) \geq 0$  for all  $Y$ , we have that  $\Delta G_{t-1}$  is non-increasing and  $G_{t-1}$  is concave.

■

The lemma thus implies the optimal policy for the Delayed Production Strategy.

**Corollary 2.2** *Given a vector of prices,  $P$ , there exists an optimal policy for the Delayed Production Strategy with an optimal order-up-to level ( $Y_t^*$ ) and an optimal save-up-to ( $S_t^*$ ).*

Thus, the structure of the Delayed Production problem combined with the limits on production capacity leads to a modified order-up-to policy. If there is sufficient capacity then produce enough to bring the inventory level up to the order-up-to level, otherwise produce to maximum capacity. Similarly, the optimal policy for inventory to save for the future is to set aside the save-up-to amount if possible, otherwise set aside the maximum available.

More importantly, the lemma also implies that the assumption of when to reject customers, either before demand is observed or after it is observed, has no impact on the optimal inventory policy. Indeed, the concavity of  $J_t(I)$  implies the following.

**Corollary 2.3** *Given a vector of prices,  $P$ , assume that the decision to reject or accept customers is made **after** the period demand has been observed. There exists an optimal policy for the Delayed Production Strategy characterized by an order-up-to level ( $Y_t^*$ ) and an optimal save-up-to ( $S_t^*$ ).*

Thus, knowing the amount of observed demand does not have any impact on how much inventory the firm should transfer for future periods.

## 2.2.2 Heuristic for Prices and Analysis

In this section, we address the problem of choosing a price vector  $P$  for the Delayed Production strategy. In many cases, a deterministic problem can provide a starting point for solving a problem with uncertainty; our strategy is to use a deterministic Pricing Problem to generate a pricing vector. Of course, this pricing vector is not necessarily the optimal one,

and thus we are also interested in the worst-case performance of this strategy relative to the optimal strategy.

The deterministic pricing problem that we analyze is similar to the stochastic pricing problems except for the relationship between demand and price. For this problem, we assume that demand is a non-increasing function of the product's price and that the relationship between demand and price is known. Since there is a known relationship between price and demand, it is sufficient to decide on sales in a particular period in order to determine the price in that period. This implies that the revenue function,  $R_t$ , can be expressed as a function of satisfied demand,  $D_t$ . Thus,  $R_t(D_t) = P_t D_t$  and the pricing problem with deterministic demand, referred to as Problem  $PP$ , can be formulated as:

$$\begin{aligned}
 (\text{PP}) \quad & \max \quad \sum_{t=1}^T (R_t(D_t) - h_t I_t - k_t X_t) \\
 & \text{subject to} \quad I_o = 0 \\
 & \quad I_{t+1} = I_t + X_t - D_t, \quad t = 1, 2, \dots, T \\
 & \quad X_t \leq q_t, \quad t = 1, 2, \dots, T \\
 & \quad I_t, X_t, D_t \text{ integer} \geq 0, \quad t = 1, 2, \dots, T.
 \end{aligned}$$

The objective is to determine prices in each period and the production schedule for the horizon in order to maximize profit. The first constraint requires that the initial inventory for the first period is zero. The second constraint ensures that inventory is balanced, and enough units are produced to meet customer sales. Production is limited by the capacity of the system in the third constraint, and the last requirement is that product be demanded or produced in integer units.

We consider now the problem of setting the price vector for the Delayed Production algorithm, using the deterministic problem. For any price  $P$ ,  $\bar{d}_t(P)$  is the corresponding expected demand at period  $t$ , and  $\bar{d}(P)$  is the vector of expected demands for price  $P$ .

To apply Problem  $PP$ , we must relate the revenue function,  $R_t(D_t)$  to the expected demand curve,  $\bar{d}_t(P)$ . Since even in the deterministic case, realized demand may be less than observed demand as generated by the demand curve, we let  $R_t(D_t) = \max_{P_t} \{D_t P_t : \bar{d}_t(P_t) \geq D_t\}$ .

We refer to Problem  $PP$  with this revenue function as  $PP(\bar{d})$ .

- *Heuristic to Choose Prices for Delayed Production:*

Solve problem  $PP(\bar{d})$ , and denote the resulting optimal profit as  $F^*(\bar{d})$ . Let  $P^H$  be the optimal prices resulting from a solution of  $D^H, X^H$ . Apply prices  $P^H$  to the Delayed Production Strategy.

We refer to the Delayed Production Algorithm with prices chosen as above as the Delayed Production *Heuristic*. To examine the effectiveness of the Delayed Production Heuristic, we analyze its worst case performance. For this purpose, let  $Z_1^*$  be the expected profit of an optimal policy for the Delayed Production Strategy, and  $Z_1^H$  be the expected profit of the Delayed Production Heuristic.

We begin by showing that the optimal solution value of Problem  $PP(\bar{d})$  is an upper bound on  $Z_1^*$ . We have:

**Lemma 2.4**  $Z_1^H \leq Z_1^* \leq F^*(\bar{d})$ .

For the proof of this result the reader is referred to Section 6.1 in the appendix. Unfortunately, the following example shows that the bound in general may be quite weak.

**Example 2.5** Consider a one period problem with no holding or production cost. There is one possible price, 1, with distribution function  $\psi^1(0) = 1 - 1/n$  and  $\psi^1(n) = 1/n$ . The available production capacity is 1 unit. Then  $P^H = 1$  with  $F^*(\bar{d}) = (1)(1) = 1$ . Of course, the optimal price is also 1, and  $Z_1^* = Z_1^H = 0 + (1)(1)(1/n) = 1/n$ . Thus, in this case,  $F^*(\bar{d})$  is an infinitely bad upper bound.

Observe that in this example, the heuristic chose the optimal price – the expected profit of the heuristic was equal to the expected profit of the optimal solution for the stochastic problem. The following example shows that in general this is not true.

**Example 2.6** Consider a one period model with no holding cost or production cost. There are 2 possible prices: a price of 1 has the distribution function  $\psi^1(0) = 1 - 1/n$  and  $\psi^1(2n) = 1/n$ , a price of  $2 - e$  has the distribution function  $\psi^{2-e}(1) = 1$ . The production capacity is limited to 2 units. In this case,  $P^H = 1$  with  $F^*(\bar{d}) = (1)(2) = 2$  and  $Z_1^H = 1(2)(1/n) = 2/n$ . However, the optimal price =  $2 - e$  with  $Z_1^* = (2 - e)(1)(1) = 2 - e$ .

So in general, the heuristic may not be effective. But observe that in both of these examples, demand has a unique structure: either zero or very high, much higher than the capacity. The challenge thus is to characterize the effectiveness of the heuristic for more reasonable demand functions. Specifically, we provide a data dependent worst-case performance analysis for the Delayed Production Heuristic.

For any price  $P$ , let  $d_t^{\min}(P)$  be the minimum possible demand at period  $t$ , and let  $\xi^{\min}(P) = \min_t [d_t^{\min}(P)/\bar{d}_t(P)]$ . Consider  $\xi^{\min} = \xi^{\min}(P^H)$ , defined over the prices generated by the Delayed Production Heuristic.

**Lemma 2.7** If the deterministic revenue curves,  $R_t(D_t)$  are concave with respect to satisfied demand, then we have  $Z_1^H \geq \xi^{\min} Z_1^*$ .

For details of the proof the reader is referred to Section 6.2 in the appendix. In the next example, we show that the bound is tight.

**Example 2.8** Consider a one period model with no holding cost but with production cost = \$2 per unit. There are only 2 possible prices,  $4 - e$  and 3, with distribution function  $\psi^{4-e}(2) = 1$ , and  $\psi^3(4\alpha) = \psi^3(4 + 4(1 - \alpha)) = \frac{1}{2}$ , respectively, for some  $\alpha$  where  $0 < \alpha < 1$ . The production capacity is limited to 4 units. Then  $P^H = 3$  with  $F^*(\bar{d}) = 4$ . Since the optimal production is  $4\alpha$  for this price, we have that  $Z_1^H = -2(4\alpha) + 3(4\alpha) = 4\alpha$ . However, the optimal price =  $4 - e$  with an optimal production quantity of 2, and  $Z_1^* = (4 - e - 2)2 = 4 - 2e$ . Hence,  $\xi^{\min} = \frac{4\alpha}{4} = \alpha$  and as  $e \rightarrow 0$ ,  $Z_1^H = 4\alpha = \alpha Z_1^* = \xi^{\min} Z_1^*$ .

### 2.2.3 Special Case: Fixed Pricing Strategy

Although many firms may be interested in finding the optimal fixed price over an entire horizon, the problem of determining the best fixed price while allowing production flexibility under *stochastic* demand has received little attention in the literature. Indeed, the problem is a special case of the Delayed Production Strategy, except that *only a single price is allowed in the horizon*. Let  $P^{Fix}$  represent the single price over the horizon, and let the price in each period  $P_t = P^{Fix}$ . The problem of determining an effective production strategy is defined as in the Delayed Production Strategy. Of course, the dynamic program described earlier holds for any price vector and therefore for a Fixed Price policy.

Our method for solving this problem follows directly from the Delayed Production Strategy. The steps are straightforward: Set a price  $P^{Fix}$  for the entire horizon; find the optimal production and inventory policy for this price; choose the price  $P^{Fix*}$  that maximizes the expected profit. Along with the optimal price, the strategy defines order-up-to and save-up-to policies that determine production and sales decisions.

### 2.3 Delayed Pricing Strategy

Many firms need to plan production in advance but would like to take advantage of price flexibility. Price flexibility is likely to provide a mechanism for clearing inventory from previous periods. As before, we define the problem using the initial inventory as the state of the system, and again the demand distribution,  $d\Psi_t^{P_t}$ , depends on the price in period  $t$ .

As before, let  $Y_t$  be the inventory level after production and before demand is realized, and  $S_t$  is the inventory that is set aside to satisfy demand in future periods. Again, we begin by assuming the decision on the amount of inventory to save for future periods is determined at the beginning of a period before demand is realized. Let  $J_t(I)$  represent the expected profit from the beginning of period  $t$  until the end of the horizon with an initial inventory level of  $I$ ; let  $G_t(Y)$  represent the expected profit-to-go with  $Y$  units of product available.

Given a vector of production values, the optimal expected profit in period  $t$  for Delayed Pricing is:

$$J_t(I) = -k_t(Y_t - I) + G_t(Y_t), \quad (3)$$

where

$$G_t(Y_t) = \max_{0 \leq S_t \leq Y_t, p_t^{min} \leq P_t \leq p_t^{max}} \left\{ \int P_t(\min(D_t, Y_t - S_t)) d\Psi_t^{P_t}(D_t) - h_{t+1}S_t - \int h_{t+1}(\max(0, Y_t - S_t - D_t)) d\Psi_t^{P_t}(D_t) + \int J_{t+1}(S_t + \max(0, Y_t - S_t - D_t)) d\Psi_t^{P_t}(D_t) \right\}. \quad (4)$$

When  $t$  is the last period in the horizon, the last term in the equation is replaced with the expected salvage value of leftover inventory given by  $\int v(\max(0, D_T - (Y_T - S_T))) d\Psi_T^{P_T}(D_T)$ .

As before, the first term in (3) is production cost, and the first term in (4) is the rev-

enue, where sales is the minimum between realized demand and the available inventory after production in period  $t$ . The next two terms in (4) represent the inventory holding cost, and finally, the profit for the future is accounted for in the last term of (4), as a function of the inventory that is carried forward.

In the next section, we discuss properties of the Delayed Pricing problem and a solution method.

### 2.3.1 Analysis of Delayed Pricing for a Given Production Schedule

Given an initial production schedule  $X$ , the Delayed Pricing Strategy becomes a question of determining a price in each period ( $P_t$ ) and the optimal amount of inventory to set aside for the future ( $S_t$ ). Unfortunately, this problem does not have the same properties as the Delayed Production Strategy, and the problem is more difficult to solve. Specifically, a result similar to Theorem 2.1 does not hold, i.e., the expected profit-to-go function in period  $t$ ,  $J_t(I)$  (equation (3)) is not concave with  $I$ .

This also implies, and unlike the Delayed Production model, that deciding when to reject customers, either before the period demand is realized or after, can have an impact on expected profit. This is shown in the next property.

**Property 2.9** *The expected profit-to-go function  $J_t(I)$  is not concave with  $I$ ; the optimal amount of inventory to save for the future,  $S_t$ , depends on the realized demand in period  $t$ .*

**Example 2.10** *Consider only the last two periods of a multi-period problem. The price in the second to last period is fixed to be \$.45 and the inventory level after production for that period is 8 units. There are two possible prices in the last period: \$1 with demand 3 or 7 with equal probabilities, and \$1.4 with demand 1 or 5 with equal probabilities. All other costs are equal to zero. Demand in period  $T - 1$  can take values between 0 and 8. Table 1 shows profit and price results for various scenarios.*

*Observe that  $J_T(I)$  is not concave, the marginal value of this function decreases (from 1.4 to 0.7) and then increases (to 0.9). Also observe that in period  $T - 1$ , the amount of inventory to save for period  $T$  depends on demand. For instance, if demand in period  $T - 1$  is 2 units, it is better to sell one unit and save 7 for period  $T$  than to sell two units in  $T - 1$ . This is true since as the table shows in the former case the expected profit is 5.45 while in the latter it is 5.4. On the other hand, if demand is three or more, the optimal policy is to sell three and save 5 units for the future for an expected profit of 5.55.*

The example shows that for Delayed Pricing, deciding on the amount to save for the future **before** demand is observed yields an inferior performance relative to determining the amount to save for the future after observing customer demand.

Now consider the optimal pricing policy for Delayed Pricing. Intuition suggests that as the initial inventory level,  $I_t$ , increases, the corresponding optimal price should decrease, since price is often used as a market clearing tool. Unfortunately, this property may not hold in general as is illustrated in the following.

Init Inv in T	$J_T(I)$ given $P = 1.4$	$J_T(I)$ given $P = 1$	$J_T(I)$	Amt Sold in T-1	Profit in T-1	$G_{T-1}(8)$ for Amt Sold in T-1
0	0	0	0	8	3.6	3.6
1	1.4	1	1.4	7	3.15	4.55
2	2.1	2	2.1	6	2.7	4.8
3	2.8	3	3	5	2.25	5.25
4	3.5	3.5	3.5	4	1.8	5.3
5	4.2	4	4.2	3	1.35	5.55
6	4.2	4.5	4.5	2	.9	5.4
7	4.2	5	5	1	.45	5.45
8	4.2	5	5	0	0	5

Table 1: Expected Profit and Optimal Price by Realized Demand

**Property 2.11** *In a period  $t$ , the optimal price in that period,  $P_t$ , does not always increase with increasing initial inventory.*

**Example 2.12** *Consider the last period in a planning horizon and assume that 2 prices are available,  $P_1$  and  $P_2$ . For price  $P_1$ , demand is either 1 or 3 with equal probability. Let  $P_2 = P_1/1.3$ , and demand is either 2 or 4 with equal probability. The following table shows the possible revenues and optimal price based as a function of the inventory available at the beginning of the period.*

Inventory	Revenue for $P_1$	Revenue for $P_2$	Optimal Price
1	$P_1 = 1.3P_2$	$P_2$	$P_1$
2	$1.5P_1 = 1.95P_2$	$2P_2$	$P_2$
3	$2P_1 = 2.6P_2$	$2.5P_2$	$P_1$
4	$2P_1 = 2.6P_2$	$3P_2$	$P_2$

Table 2: Revenue and Optimal Price by Inventory Level

*It is clear in this case that price does not always decrease with increasing inventory. Thus, the property relating price and inventory does not hold in general.*

This observation is similar to the one made by Polatoglu et al. (2000) but different than a property identified by Federgruen and Heching (1999) and is due to the fact that the expected profit-to-go function in our case is not concave.

Of course, the implication of this observation is that it is necessary to search over all prices in each period to find the price that maximizes expected profit.

### 2.3.2 Heuristic for Delayed Pricing

In this section, we suggest an heuristic for the Delayed Pricing Model. Given a production schedule, the dynamic program represented by equations (3) and (4) generates an optimal solution. Since the state space is large, and since determining discretionary sales prior to demand realization may not be the best strategy, we force  $S_t = 0$  for all  $t$ , see Appendix 6.3.2 for the algorithm.

To determine a production schedule we use a deterministic approximation of the demand curves. For this purpose, we use Problem  $PP$ , as outlined in Section 2.2.2.

- *Heuristic to Choose Production Quantities for Delayed Pricing*  
Solve Problem  $PP(\bar{d})$  as defined in Section 2.2.2. Let the optimal production quantities be  $X^H$ . Apply the production schedule  $X^H$  to equations (3) and (4).

We refer to the Delayed Pricing Algorithm with the production schedule as determined above as the Delayed Pricing Heuristic.

## 3 Computational Analysis

In this section, we report on an extensive computational study conducted to develop insights about the benefits of dynamic pricing and partial planning strategies. We compare the solutions obtained from the Delayed Production, Delayed Pricing, and Fixed Pricing strategies. Our goal is three-fold: (i) examine the performance of fixed pricing under stochastic demand (ii) identify the situations where dynamic pricing provides significant increases in profit compared to fixed pricing in a stochastic demand setting, and (iii) determine if one partial planning strategy is preferred over the other.

### 3.1 Computational Details

#### 3.1.1 Performance Measures

To examine the performance of Fixed Pricing under stochastic demand, we consider two strategies for generating a fixed price. For the first, we define a problem called the Deterministic Fixed Pricing problem ( $DFP$ ). Problem  $DFP$  requires a single price for all periods and integer demands; assumptions are the same as for the deterministic pricing problem  $PP$ . The optimal solution to this deterministic problem, which can be solved with a non-linear integer solver available at NEOS, provides a fixed price policy, and it represents the process used by many firms to choose a fixed price strategy. Using this price, we determine the optimal production and inventory policy by applying the Delayed Production Algorithm, we refer to this stochastic strategy as the  $SDFP$  strategy. The second strategy is based on finding the optimal fixed price taking uncertainty into account. We refer to this problem as problem  $FP$ , or the Fixed Pricing Strategy.

To compare the *SDFP* and *FP* strategies, we define the *profit potential* as

$$\text{Profit Potential} = \frac{Z_{FP}}{Z_{SDFP}} - 1,$$

where  $Z$  indicates the expected profit of the problem being solved. Similarly, we measure the profit potential of either Delayed Production or Delayed Pricing relative to the best fixed price strategy, i.e., strategy *FP*.

### 3.1.2 Parameters and Case

In the computational study, we examine cases with variations on the following parameters: capacity,  $q_t$ ; demand,  $D_t$ ; holding cost,  $h_t$ ; and production cost,  $k_t$ . Each problem consists of ten periods unless stated otherwise. Except in the cases indicated, the holding cost is a constant value for each period, and there is no production cost. The salvage value was taken to be a salvage cost, equal to twice holding cost in the last period. The demand curve was determined by aggregating linear demand curves for sixty-five similar products provided to us by our industrial partner. We refer to this demand curve as the *standard demand curve*.

Based on feedback from our industrial sponsor, we model demand curves to be a linear function of price, based on the following three parameters: the base price  $p^{base}$ , the base demand or volume  $v^{base}$ , and the demand elasticity,  $E$  and are given by

$$V^{new} = v^{base} + E * (v^{base} / p^{base}) * (P^{new} - p^{base}),$$

where  $V^{new}$  and  $P^{new}$  are the actual demand and price, respectively. The demand elasticity is defined as the percentage change in volume / percentage change in price, and it is assumed that the elasticity is the same for a given demand curve over the range of demands considered. The demand curve as provided can easily be converted into the traditional form with slope and intercept, however the description of the parameters examined is more meaningful with the original form of the curve.

We are particularly interested in the effect of demand variability on the performance of dynamic and fixed pricing. We consider both seasonality (changes in the deterministic demand curve) as well as uncertainty (the epsilon component of the demand function).

To measure *demand uncertainty*, we define the coefficient of variation of demand in a given period as follows:  $CV_U = stdev(D_t) / mean(D_t)$ , where  $D_t = d_t(P_t, \epsilon_t(P_t))$ . For a given test case, the demand uncertainty is the same in each period. Thus, the random component,  $\epsilon_t$ , can be calculated as a function of the demand uncertainty, and the value depends on the price in each period. Alternatively, we also tested cases where  $\epsilon$  was independent of price and determined the demand uncertainty in each period, and we obtained similar results for these tests.

To measure *demand seasonality*, we are interested in the variability of the deterministic portion of demand over the entire time horizon. Thus it is useful to define the *optimal uncapacitated demand*, which is denoted as  $Dem^*$  in the figures we show. This value is the

optimal sales if there were no production capacity limits. To measure seasonal variability in deterministic demand, we use the coefficient of variation of  $Dem^*$ ; we also refer to this as  $CV_S$ , where the mean and standard deviation of the deterministic ratio are calculated over the entire horizon.

Capacity levels, while kept constant for a particular problem, were allowed to vary from problem to problem at .5, .75, and 1.0 times the uncapacitated optimal demand of the standard linear curve. Below we describe the specifics of the cases used.

- *Case 1: Effect of Demand Seasonality*

We study the impact of demand seasonality by keeping the demand uncertainty constant over all periods ( $CV_U = 0.2$ ). Variation in demand was introduced in the demand curve by allowing both the base demand and the demand elasticity to change periodically. For this purpose, we start with a given base demand and demand elasticity, which are referred to as the standard parameters. We then vary the base demand and demand elasticity periodically as a percentage of the standard values. The percent variation levels considered are 0.8 - 1.2, 0.6 - 1.4, 0.4 - 1.6, and 0.2 - 1.8 for the base demand and 0.9 - 1.1, 0.8 - 1.2, 0.7 - 1.3 and 0.6 - 1.4 for the demand elasticity. The variation in demand elasticity is smaller than the variation in base demand to ensure that elasticities fell in acceptable ranges. In all cases, the average base demand and the average elasticity across the entire horizon were equal to the standard values. The resulting values of  $CV_S$  are .08, .17, .26, and .37.

- *Case 2: Effect of Demand Uncertainty*

Initially we consider a case with zero demand seasonality, where the deterministic demand curve in all periods is equal to the standard demand curve. We vary the demand uncertainty by fixing  $CV_U$  to be 0.1, 0.2, 0.3 and 0.4 (and calculating a corresponding standard deviation of  $\epsilon$  in each period). The bounds on the stochastic portion of demand are 2 standard deviations from the mean in either direction, or 0, whichever is greater. For the combined effect of seasonality and uncertainty, we fixed the seasonality to be the second case described above ( $CV_S = 0.17$ ) and consider the same levels of  $CV_U$ .

We were also interested in other cases, particularly ones useful in comparing different partial planning strategies. To gauge this effect, we consider the following:

- *Case 3: Effect of Production Cost*

We obtained an average production cost from our industrial partner; we call this the *standard* production cost. Additionally, we examined various levels of production cost, 0, 0.5, 1.0, and 1.5 times the standard production cost. For these cases we let  $CV_U = 0.2$  and  $CV_S = 0.17$ ; we considered this to be our standard case of demand variability.

- *Case 4: Effect of Product Characteristics*

The cases described thus far have all had the same basic seasonality factors. We were

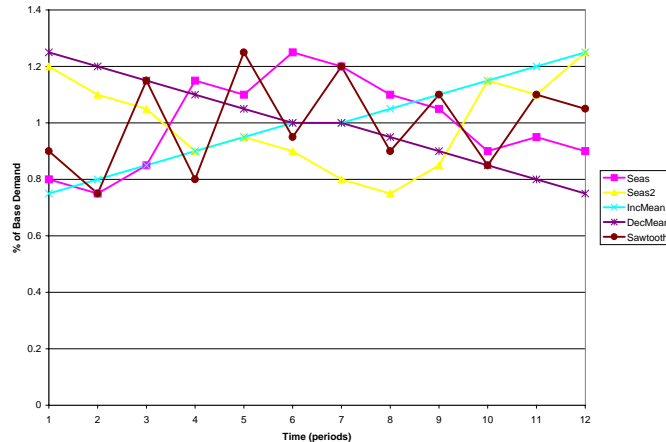


Figure 1: 5 Examples of Product Seasonality

also interested in the performance of the various strategies under other seasonality factors. The seasonality factors that were used loosely correspond with different product types. A brief description of the product types follows, with a full description available in Biller et al (2000). Decreasing Mean (DecMean) has demand that is non-increasing over the horizon, like demand for a laptop. Increasing Mean (IncMean) has demand that is non-decreasing over the horizon, similar to products with word-of-mouth effects, such as a musical CD. Seasonality 1 (Seas1) has quarterly seasonality low at first, higher in the middle, low at the end of the horizon, and Seas2 is the scenario corresponding to opposite quarterly demand levels. The sawtooth scenario is used for additional comparison, and has demand alternating between low and high, with some additional variability over the horizon. Figure 1 presents the optimal uncapacitated deterministic demand of each scenario described. In all cases presented, the  $CV_S$  is 0.12 and the  $CV_U$  is 0.2, the planning horizon has twelve periods, and the production cost is standard for the demand curve used.

The complexity of the Partial Planning algorithms can be decreased by employing strategies that exploit the structure of the problem. In particular, a Fast Fourier Transform (FFT) may be used to calculate a "convolution" which is inherent in the calculations above. For further details on this, see Swann (2001).

## 3.2 Insights from Computational Analysis

Our computational based case study reveals a number of insights. In particular, it shows that:

- Partial planning dynamic pricing significantly increases profit in many situations, particularly when deterministic demand is highly variable and when capacity is tight.
- Performance of dynamic pricing tends to increase as seasonality increases and as capacity is tighter.
- The Delayed Pricing Heuristic usually outperforms the Delayed Production Heuristic. Exceptions include when production cost is high.

Below we discuss these in more detail.

### 3.2.1 Performance Trends

- *As capacity becomes more constrained, the benefit of dynamic pricing tends to increase.* This effect can be significant. For example, see Figure 2, where uncertainty is fixed and seasonality increases. In a case with moderate variability ( $CV_S = 0.26$ ), constraining the capacity to 50% leads to 6.5% profit potential for the Delayed Production Heuristic as compared to 1.4% for the case with capacity equal to  $Dem^*$ . This holds true when examining uncertainty as well (Figure 3 with zero seasonality and Figure 4 with fixed seasonality). For instance, in Figure 4 a low variability case ( $CV_U = 0.1$ ) for the Delayed Pricing Heuristic has about 1% profit for the case with large available capacity and almost 5% profit for the case with tight capacity constraints. This supports trends seen in the computational analysis with deterministic demand as well.

Of course, this trend is not surprising, since it becomes more important to allocate resources effectively when those resources are scarce. Both of the partial planning strategies have pricing as a flexibility tool while the Fixed Pricing strategy does not.

- *The benefit of dynamic pricing tends to increase as demand seasonality increases.* Examine now Figure 2. In this graph we see the trend for both Delayed Production and Delayed Pricing. For the Delayed Pricing Heuristic, with the middle level of capacity, the a low seasonality ( $CV_S = 0.08$ ) sees 2.2% increase in profit, whereas the high level of seasonality has 6.1% profit potential. The trend holds true for cases with higher capacity as well; a low seasonality ( $CV_S = 0.08$ ) has 0.3% profit potential while the highest seasonality has 1.9% profit potential.

This is somewhat intuitive. One expects that as the level of variability increases, the need for flexibility increases, and this flexibility is provided by the non-stationary pricing strategies.

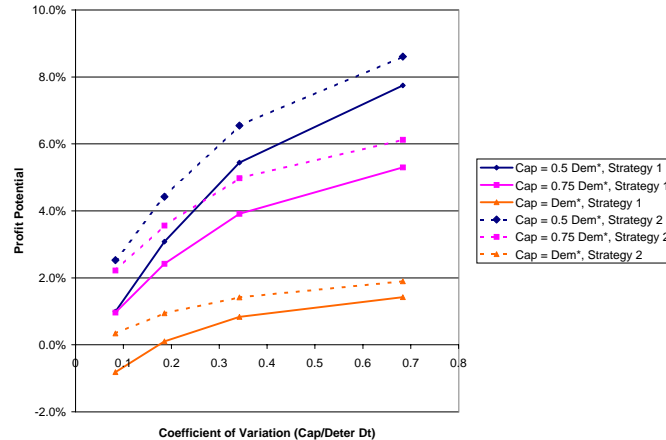


Figure 2: Impact of Seasonal Variability when Uncertainty is Constant

In contrast, the performance of the strategies under increasing *uncertainty* is less clear. Figures 3 and 4 depict the effect of demand uncertainty when seasonality is zero or constant over the horizon. In Figure 3, the profit potential for the Delayed Pricing Heuristic increases with increasing uncertainty, but the Delayed Production Heuristic performs more poorly as the uncertainty increases. This likely points to an inability of Delayed Production to deal with unpredictable inventory resulting from large variability. There are a number of cases where flexibility in price may be a more effective tool than production flexibility since production cannot truly be used to *reduce* the inventory currently in the system. In Figure 4, a similar trend holds—the Delayed Production Heuristic has decreasing performance as stochastic uncertainty increases. The performance of the Delayed Pricing Heuristic is somewhat better than Delayed Production, and this performance is fairly flat over the different cases. Of course, this graph is affected by the type and level of deterministic demand seasonality.

It is also useful to note that while the Delayed Pricing Heuristic outperforms Fixed Pricing (and the Delayed Production Heuristic) in all of the cases shown, at times Fixed Pricing is a better policy than the Delayed Production Heuristic. This is particularly true in the case with zero seasonality (Figure 3). Of course, this is due to the fact that in the fixed price policy we use the optimal price, whereas in the Delayed Production Heuristic the prices are not optimally chosen.

A slight effect, see Figure 5, is that the benefit from dynamic pricing (Delayed Production and Delayed Pricing) may increase as production cost increases. This is not necessarily a strong effect; indeed, in one case it is clearly not true (Delayed Pricing, large capacity available). But in many of the other cases, a higher production cost led to more opportunity

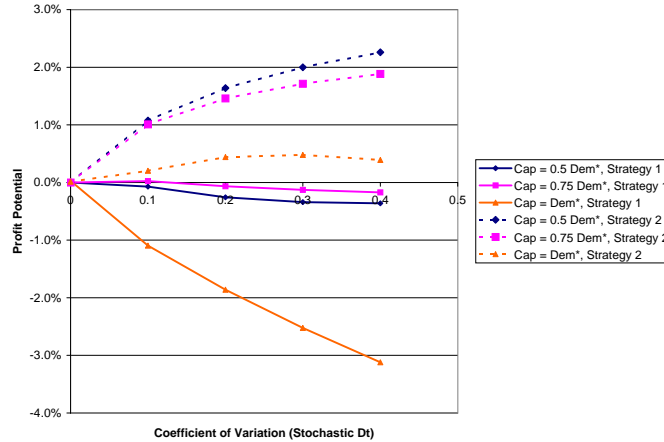


Figure 3: Impact of Uncertainty when Seasonality is Zero

for dynamic pricing strategies.

We also examine the performance of the Delayed Production Heuristic compared to the upper bound developed in Lemma 2.4.

- *Delayed Production achieves a significant portion of the Upper Bound*

In general, the Delayed Production Heuristic performs quite well compared to the upper bound. In all cases, the expected profit of the Delayed Production Heuristic was in the range of 92% to 99% of the expected profit of the upper bound of Lemma 2.4. In most cases, the ratio of expected profit generated by the heuristic to the upper bound, i.e.,  $Z_1^H / F^*(\bar{d})$ , is around 96%. Typically, the performance of expected profit compared to the upper bound tended to decrease with increasing uncertainty. This is not surprising, since the bound does not account for uncertainty at all and is likely to be weaker with high uncertainty.

A brief discussion of the save-up-to inventory targets is also useful. As described previously, these inventory targets were used in the determination of the Delayed Production Heuristic. In most cases, we found that the optimal value of the target was zero. However, in some cases the optimal solution resulted in non-zero values for the save-up-to targets.

In particular, one situation that may result in non-zero values of the save-up-to targets is when production cost is high. For example, in Figure 5, the targets resulting from the Delayed Production Heuristic were non-zero for at least one period when the production cost is at the highest level. Another case that may lead to non-zero save-up-to values is when stochastic seasonality is moderate ( $CV_U = 0.2$ ) and deterministic seasonality is high

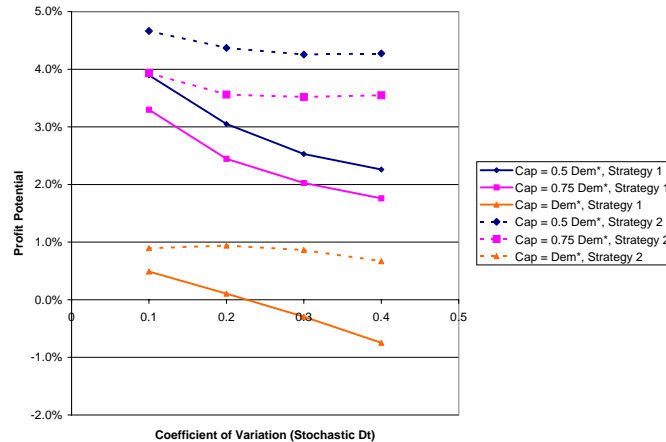


Figure 4: Impact of Uncertainty when Seasonality is Constant

( $CV_S = .37$ ), see Figure 2. This is not surprising, since the deterministic seasonality values are known in advance of production and inventory decisions. In contrast, when stochastic seasonality is high, see Figure /reffig:zero seas, none of the cases depicted had non-zero save-up-to targets. Finally, for the cases that have both deterministic and stochastic demand variability, shown in Figure 4, the cases with  $CV_U = 0.1$  result in some non-zero target values.

For the cases described with positive save-up-to values, these values had minimal effect on expected profit. This was ascertained by applying the Delayed Production Heuristic with save-up-to values of 0 and comparing the expected profit.

### 3.2.2 Comparison of Strategies

- *Making fixed pricing decisions under deterministic demand is often sufficient.*

Intuition suggests that making a fixed pricing decision while considering stochastic demand would be a significant improvement over making the decision under deterministic demand. However, for most of the cases tested, the *FP* strategy did not offer significant improvement over the *SDFP* strategy described above, e.g., as little as one half of one percent.

An example of a situation where the *FP* strategy did make a significant improvement in profit over *SDFP* is when the deterministic seasonality was zero and capacity was high. For these cases, the profit potential ranged from 1% to 3%, see Figure 6. Further testing could indicate additional scenarios in which Fixed Pricing does offer significant

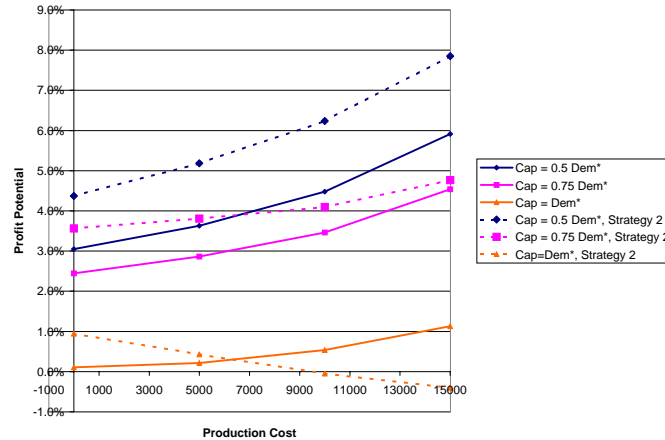


Figure 5: Impact of Production Cost

improvement.

Finally, the following insight is also suggested by Figure 7 and the previous analysis.

- *Delayed Pricing outperforms Delayed Production most of the time.* Indeed, in almost all of the cases shown, Delayed Pricing provided a higher increase in profit than Delayed Production. For instance, in a fairly realistic case in Figure 4, where  $CV_U = 0.2$ , the profit increase from Delayed Pricing (Strategy 2) is higher by about one percentage point at all capacity levels.

However, there are exceptions. In particular, when production cost and capacity levels are high, Figure 5 shows that Delayed Production can outperform Delayed Pricing, by more than one percentage point. In this case the Delayed Production Heuristic had non-zero save-up-to values, however, these values had an insignificant effect on the expected profit of the Delayed Production Strategy. The improvement of Delayed Production over Delayed Pricing is also seen when the basic demand seasonality is of different types, and in this case none of the save-up-to values were positive, see Swann (2001) for this analysis.

The result concerning the performance of Delayed Pricing versus Delayed Production fits in with earlier results obtained on a somewhat related two stage problem considered by Van Mieghem and Dada (1999). They consider a single period model where the postponed decisions are made after demand is realized. They found that in many instances Pricing

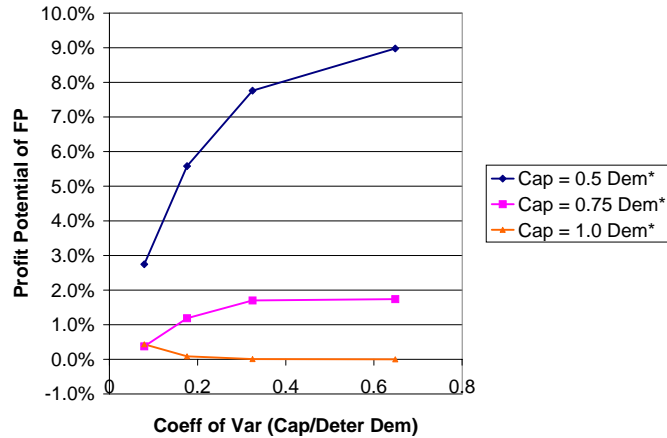


Figure 6: Comparison of Expected Profit for FP and DFP Strategies

Postponement outperformed Production Postponement; one exception was in a case with high production cost.

## 4 Extensions and Conclusions

In this paper, we present a number of partial planning strategies, motivated by the variety of problems that firms must address. For all of the strategies, Delayed Production, Delayed Pricing, and Fixed Pricing, we provide methods to determine pricing and production decisions. We introduce the general concept of discretionary sales, where inventory is set aside to satisfy future demand even if sales are lost in the current period.

We analyze the strategies presented to obtain insights for decision makers. In the case of Delayed Production (and thus Fixed Pricing as well), we show that the structure of the problem leads to an optimal policy characterized by two parameters; an order-up-to level,  $Y_t$  and a save-up-to level,  $S_t$ . If the inventory level at the beginning of the period,  $I_t$ , is below  $Y_t$ , an order of size  $\min\{Q_t, Y_t - I_t\}$  is placed; otherwise no order is made. Interestingly, the amount to save for the future,  $S_t$ , is independent of realized demand and thus can be determined at the beginning of the planning horizon. That is, the decision of how many customers to reject, in order to save inventory for future periods, can be done regardless of the volume of demand in a particular period.

Computational analysis allowed us to provide additional insights about the partial planning strategies and their relative performances. For example, choosing the optimal fixed price does not necessarily offer significant benefit compared to selecting a fixed price through a

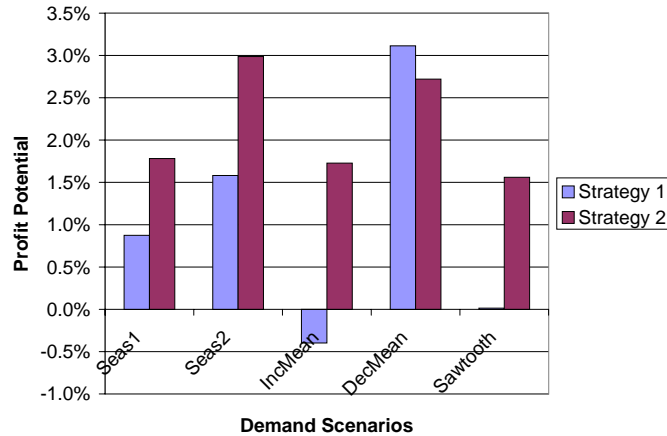


Figure 7: Impact of Type of Seasonal Variability

deterministic approximation of the demand curves.

Generally, dynamic pricing strategies, i.e., Delayed Pricing and Delayed Production, outperformed Fixed Pricing, and this difference often increased with increasing seasonality or tightening capacity. In particular, in our heuristics, Delayed Pricing generally had higher expected profit than Delayed Production. This may be explained as follows. The advantage the Delayed Pricing policies have is the ability to decrease or increase demand depending on inventory and capacity levels. This is not the case in Delayed Production. Here, the decision maker has only the ability to build inventory. However, when inventory levels are high, Delayed Production has no mechanism that allows it to generate more demand.

There are a number of future directions suggested by the results in this paper and by industry motivated situations. Further exploration of production and inventory models where inventory may be set aside for the future would be useful. Of significant interest is the problem with multiple products and multiple parts, where there is limited supply of parts and limited common production capacity. We are also interested in the problem where a firm has full flexibility on price and production decisions and makes both of these decisions in each period before customer demand is realized. Finally, the problems that we have analyzed are all under the assumption that the delayed decision is made before customer orders are received. It would be useful to analyze Postponement Strategies as described by Van Mieghem and Dada (1999) in a multi-period setting, where the secondary decision is made after demand is realized, and compare the two types of planning strategies. Of course, the value of the demand information should improve the decision making in the Postponement Strategies compared to the Partial Planning Strategies.

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## 6 Appendices

### 6.1 Proof of Lemma 2.4

In this section of the appendix we prove that the performance of the Delayed Production Strategy ( $Z_1^*$ ) has an upper bound of the profit from a deterministic pricing problem using expected demand ( $F^*(\bar{d})$ ). We begin by defining an additional deterministic pricing problem, and we show additional lemmas based on this deterministic problem that lead to the final upper bound.

For any given price vector  $P$  and realized demand vector  $d$ , the problem of finding the production quantities,  $X$ , and sales quantities,  $D$ , that maximize profit under deterministic demand can be written as:

$$\begin{aligned}
\text{SP}(P,d): \quad \max \quad & F^{P,d}(D) = \sum_{t=1}^T [R_t^{P,d}(D_t) - h_t I_t - k_t X_t] \\
\text{subject to} \quad & I_0 = 0 \\
& I_{t+1} = I_t + X_t - D_t, \quad t = 1, 2, \dots, T \\
& X_t \leq q_t, \quad t = 1, 2, \dots, T \\
& I_t, X_t, D_t \text{ integer } \geq 0, \quad t = 1, 2, \dots, T.
\end{aligned}$$

with

$$R_t^{P,d}(D_t) = \begin{cases} P_t D_t & \text{for } D_t \leq d_t \\ P_t d_t - (D_t - d_t) & \text{for } D_t > d_t. \end{cases}$$

and other parameters are as described previously. Observe that if  $D_t > d_t$ , then the revenue function includes a penalty proportional to  $(D_t - d_t)$ . Thus the optimal solution for  $SP(P, d)$  is such that  $D_t \leq d_t$ . Other choices of penalty functions are also possible; in all cases,  $R_t$  should be a decreasing function of  $D_t$ .

Let  $LSP(P, d)$  be the problem corresponding to the linear relaxation of problem  $SP(P, d)$ . Let  $\tilde{F}^{P,d}$  be the optimal objective function value of problem  $SP(P, d)$ , and  $\tilde{X}^{P,d}, \tilde{D}^{P,d}$  be the corresponding optimal production schedule and demand to be satisfied.

As before, for any price  $P$ , let  $\bar{d}_t(P)$  be the corresponding expected demand at period  $t$ , and let  $\bar{d}(P)$  be the corresponding vector of demand curves. Let  $E_d(\tilde{F}^{P,d})$  be the expected profit for Problem  $SP$  for a price  $P$  over all corresponding realized demands  $d$ .

We now show an upper bound from Problem  $SP(P, d)$  with expected demand:

**Lemma 6.1** *For any price  $P$ ,  $\tilde{F}^{P,\bar{d}(P)} \geq E_d(\tilde{F}^{P,d})$ , i.e., the optimal profit for Problem  $SP(P, d)$  with expected demand is an upper bound on the expected profit for any price  $P$  over the realized demand  $d$ .*

*Proof.* For any price  $P$  and demand  $d$ , let the vector of expected values of the production, sales, and inventory variables be defined as

$$\bar{X} = E_d(\tilde{X}^{P,d}), \quad \bar{D} = E_d(\tilde{D}^{P,d}), \quad \text{and} \quad \bar{I} = E_d(\tilde{I}^{P,d}).$$

Clearly the optimal production and sales values,  $(\tilde{X}^{P,d}, \tilde{D}^{P,d})$ , are feasible for  $SP(P, d)$ :

$$\begin{aligned}
\tilde{I}_0^{P,d} &= 0 \\
\tilde{I}_{t+1}^{P,d} &= \tilde{I}_t^{P,d} + \tilde{X}_t^{P,d} - \tilde{D}_t^{P,d} \geq 0, \quad t = 1, 2, \dots, T \\
\tilde{X}_t^{P,d} &\leq q_t, \quad t = 1, 2, \dots, T \\
\text{and } \tilde{X}_t^{P,d}, \tilde{D}_t^{P,d} &\geq 0, \quad t = 1, 2, \dots, T;
\end{aligned}$$

Taking expected values, we obtain the following equations:

$$\begin{aligned}
\bar{I}_0 &= 0 \\
\bar{I}_{t+1} &= \bar{I}_t + \bar{X}_t - \bar{D}_t = E_d(\tilde{I}_t^{P,d}) + E_d(\tilde{X}_t^{P,d}) - E_d(\tilde{D}_t^{P,d}) \\
E_d(\tilde{I}_{t+1}^{P,d}) &= E_d(\tilde{I}_t^{P,d} + \tilde{X}_t^{P,d} - \tilde{D}_t^{P,d}) \geq 0, \quad t = 1, 2, \dots, T \\
\bar{X}_t &= E_d(\tilde{X}_t^{P,d}) \leq q_t, \quad t = 1, 2, \dots, T \\
\text{and } \bar{X}_t &= E_d(\tilde{X}_t^{P,d}) \geq 0, \bar{D}_t = E_d(\tilde{D}_t^{P,d}) \geq 0, \quad t = 1, 2, \dots, T.
\end{aligned}$$

Hence the set of vectors of expected values  $(\bar{X}, \bar{D})$  is a feasible solution of the problem  $LSP(P, \bar{d}(P))$ .

Since  $R_t^{P,d}(D_t)$  is decreasing for  $D_t \geq d_t$ , by the optimality of  $\tilde{D}^{P,d}$  we know that  $\tilde{D}_t^{P,d} \leq d_t$ ; hence  $\bar{D}_t = E_d(\tilde{D}_t^{P,d}) \leq E_d(d_t) = \bar{d}_t(P)$  for all  $t$ . Since  $\bar{D}_t \leq \bar{d}(P)_t$  and  $\tilde{D}_t^{P,d} \leq d_t$ , and since  $\bar{D}_t = E_d(\tilde{D}_t^{P,d})$ , then  $R_t^{P, \bar{d}(P)}(\bar{D}_t) = E_d[R_t^{P,d}(\tilde{D}_t^{P,d})]$  for all  $t$ . Thus,  $\sum_{t=1}^T [R_t^{P, \bar{d}(P)}(\bar{D}_t) - h_t \bar{I}_t - k_t \bar{X}_t] = \sum_{t=1}^T E_d [R_t^{P,d}(\tilde{D}_t^{P,d}) + h_t \tilde{I}_t^{P,d} - k_t \tilde{X}_t^{P,d}] = E_d(\tilde{F}^{P,d})$ , and the optimal objective function value of  $LSP(P, \bar{d}(P)) \geq \sum_{t=1}^T [R_t^{P, \bar{d}(P)}(\bar{D}_t) - h_t \bar{I}_t - k_t \bar{X}_t] = E_d(\tilde{F}^{P,d})$ .

Since the capacity  $Q$  and customer demand are both integer vectors,  $LSP(P, d)$  has an integral optimal solution; note that  $\bar{d}(P)$  is assumed to be integral. Hence  $\tilde{F}^{P, \bar{d}(P)}$  = optimal objective function value of  $LSP(P, \bar{d}(P)) \geq E_d(\tilde{F}^{P,d})$ .

■

Now consider a specific choice of price vector. Set price  $\underline{P}$  such that  $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} = \max_P \tilde{F}^{P, \bar{d}(P)}$ .

That is, for each possible price vector  $P$ , solve Problem  $SP(P, \bar{d}(P))$ , and we set  $\underline{P}$  equal to the prices that maximize profit for Problem  $SP(P, \bar{d}(P))$  over all possible price vectors.

Under this choice of  $P$ , the deterministic profit from Problem  $SP(P, d)$  is an upper bound on the performance of the Delayed Production Strategy under any choice of price vector, as shown in the following lemma.

**Lemma 6.2**  $Z_1^* \leq \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$ .

*Proof.* For any price  $P$ , apply the Delayed Production Algorithm, and identify the corresponding optimal order-up-to and save-up-to policies. Given a vector of realized demand  $d$ , let  $Z^1(P, d)$  be the profit resulting from following these identified policies. Then,

$$\begin{aligned}
Z_1^* &= \max_P E_d[Z^1(P, d)] \\
&\leq \max_P E_d[\tilde{F}^{P,d}] \\
&\leq \max_P \tilde{F}^{P, \bar{d}(P)}, \text{ by Lemma 6.1} \\
&\leq \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}, \text{ by the choice of } \underline{P}.
\end{aligned}$$

■

Of course, this also implies that the expected profit from the Delayed Production Heuristic is bounded according to this lemma. Indeed, the upper bound based on the specific choice of prices from Problem  $PP(\bar{d})$ ,  $F^*(\bar{d})$ , is equivalent to the upper bound developed in Lemma 6.2,  $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$ . Recall that the revenue function in  $PP(\bar{d})$  is  $R_t(D_t) = \max_{P_t} \{D_t P_t : \bar{d}_t(P_t) \geq D_t\}$ . The optimal solution is  $D^H, X^H, P^H$ , with  $F^*(\bar{d}) = F^H(D^H)$  and  $R_t^H(D_t^H) = D_t^H P_t^H$ . Thus, we have:

**Corollary 6.3**  $F^*(\bar{d}) = \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$ .

*Proof.* Since  $(D^H, X^H)$  is a feasible solution of  $SP(P^H, \bar{d}(P^H))$ ,  $D^H \leq \bar{d}(P^H)$ . This implies  $R_t^{P^H, \bar{d}(P^H)}(D_t^H) = D_t^H P_t^H = R_t^H(D_t^H)$ , and so  $F^*(\bar{d}) \leq \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$ .

Since  $R_t^{\underline{P}, \bar{d}(\underline{P})}(D_t)$  is decreasing for  $D_t \geq \bar{d}_t(\underline{P})$ ,  $\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})} \leq \bar{d}_t(\underline{P})$  by the optimality of  $\tilde{D}^{\underline{P}, \bar{d}(\underline{P})}$ . Hence  $(\tilde{D}^{\underline{P}, \bar{d}(\underline{P})}, \tilde{X}^{\underline{P}, \bar{d}(\underline{P})})$  is a feasible solution of PPD. We know that  $\tilde{D}^{\underline{P}, \bar{d}(\underline{P})} \underline{P}_t \leq D_t^H \underline{P}_t$  (by the optimality of  $D_t^H$ )  $\leq \max_{P_t} \{D_t^H P_t : D_t^H \leq \bar{d}_t(P_t)\}$  (since this

is maximized over all  $P_t$ )  $= R_t^H(D_t^H)$ . Thus,  $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} \leq F^*(\bar{d})$ , and the proof is complete. ■

Given Lemma 6.2 and Corollary 6.3, we have that  $Z_1^* \leq F^*(\bar{d})$ , and thus Lemma 2.4 holds.

### Extension: Alternative Demand Assumption

The results we have shown can also be extended to another demand scenario, in which demand in period ( $D_t$ ), may depend not only on the price in period  $t$  ( $P_t$ ), but also on the entire price vector,  $\underline{P}$ . In this case, the prices for the Delayed Production Heuristic are set as described above, specifically: Set price  $\underline{P}$  such that  $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} = \max_{\underline{P}} \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$ . The performance of the Delayed Production heuristic under this demand scenario has an upper bound of  $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})}$  as described by Lemma 6.2.

## 6.2 Proof of Lemma 2.7

The structure of Problem  $PP$  (and thus  $SP(P, d)$ ) places it into a class of problems that can sometimes be solved with a greedy algorithm, or Marginal Allocation Algorithm (MAA). The greedy algorithm applied to Problem  $PP$  would be as follows:

### Greedy Algorithm or MAA:

*Step 0:* Set  $D_t = 0$  in each period;

*Step 1:* Choose a period  $t$  to increase demand by one unit such that the contribution to profit is maximized and the feasibility of the production schedule is maintained.

*Step 2:* If no such period exists, stop.

The algorithm may also be modified to take into account lower and upper bounds on demand and price.

Billar, Chan, Simchi-Levi, and Swann (BCSS) (2002) analyzed conditions under which the greedy algorithm solves Problem  $PP$ . Specifically, their main result is the following:

**Theorem 6.4 (BCSS (2002) Main Result)** *If the revenue functions  $R$  are concave in Pricing Problem  $PP$ , then the greedy algorithm provides the optimal solution.*

When the revenue functions are not concave, other solution methods, namely non-linear optimization methods, may be used to solve the deterministic pricing problem.

In the proof below, we show that  $Z_1^H \geq \xi^{\min} Z_1^*$ , where  $\xi^{\min} = \xi^{\min}(P^H) = \min_t [d_t^{\min}(P^H)/\bar{d}_t(P^H)]$ , and  $d_t^{\min}(P)$  is the minimum possible demand at period  $t$ , as described in Lemma 2.7.

*Proof.* Apply MAA as described in Section 2.2.2 to solve  $SP(\underline{P}, \bar{d}(\underline{P}))$ , Let  $F_t(k)$  be the marginal profit of satisfying the  $k^{\text{th}}$  unit of demand at period  $t$ . Then  $F_t$  is non-increasing and  $\tilde{F}^{\underline{P}, \bar{d}(\underline{P})} = \sum_{t=1}^T \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k)$ .

Let  $D_t^{\min} = \min \left\{ d_t^{\min}(\underline{P}), \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})} \right\}$ . Since  $D^{\min} \leq \tilde{D}^{\underline{P}, \bar{d}(\underline{P})}$ , it is a feasible solution for  $SP(\underline{P}, \bar{d}(\underline{P}))$ . Consider satisfying  $D^{\min}$  in the same order  $\tilde{D}^{\underline{P}, \bar{d}(\underline{P})}$  is satisfied using MAA, we have

$$\begin{aligned}
F^{\underline{P}, \bar{d}(\underline{P})}(D^{\min}) &= \sum_{t=1}^T D_t^{\min} \sum_{k \leq D_t^{\min}} \frac{F_t(k)}{D_t^{\min}} \\
&\geq \sum_{t=1}^T \left( \frac{D_t^{\min}}{\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k) \right), \\
&\quad \text{since } \sum_{k \leq D_t^{\min}} \frac{F_t(k)}{D_t^{\min}} \geq \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} \frac{F_t(k)}{\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} \\
&= \sum_{t=1}^T \left( \min \left\{ 1, \frac{d_t^{\min}(\underline{P})}{\tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} \right\} \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k) \right), \\
&\quad \text{by the definition of } D^{\min} \\
&\geq \sum_{t=1}^T \min \left\{ 1, \frac{d_t^{\min}(\underline{P})}{\bar{d}_t(\underline{P})} \right\} \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k), \text{ since } \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})} \leq \bar{d}_t(\underline{P}) \\
&\quad \text{by the optimality of } \tilde{D}^{\underline{P}, \bar{d}(\underline{P})} \\
&\geq \sum_{t=1}^T \frac{d_t^{\min}(\underline{P})}{\bar{d}_t(\underline{P})} \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k) \\
&\geq \xi^{\min} \sum_{t=1}^T \sum_{k \leq \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})}} F_t(k), \text{ by the definition of } \xi^{\min} \text{ and} \\
&\quad \text{since } \tilde{D}_t^{\underline{P}, \bar{d}(\underline{P})} \leq \bar{d}_t(\underline{P}) \\
&= \xi^{\min} \tilde{F}^{\underline{P}, \bar{d}(\underline{P})}
\end{aligned}$$

Since  $R_t^{P, \bar{d}(P)}(D_t) = R_t^{P, d^{\min}(P)}(D_t)$  for  $D_t \leq d_t^{\min}(P)$ , then

$$\begin{aligned} \tilde{F}^{P, d^{\min}(P)} &\geq F^{P, d^{\min}(P)}(D^{\min}), \text{ by optimality} \\ &= F^{P, \bar{d}(P)}(D^{\min}), \text{ since the revenue is the same} \\ &\geq \xi^{\min} \tilde{F}^{P, \bar{d}(P)}, \text{ proof above.} \end{aligned}$$

Since  $R_t^{P, \bar{d}(P)}(D_t)$  is decreasing for  $D_t \geq d_t^{\min}(P)$ , then  $\tilde{D}^{P, d^{\min}(P)} \leq d_t^{\min}(P)$ . Hence  $(\tilde{D}^{P, d^{\min}(P)}, \tilde{X}^{P, d^{\min}(P)})$  is a feasible policy for the Delayed Production Strategy with expected profit the same as the profit from the deterministic problem,  $\tilde{F}^{P, d^{\min}(P)}$ . Since  $Z_1^H$  is the maximum expected profit for price  $= P$  obtained by following the optimal policy for  $D$  and  $X$ ,  $Z_1^H \geq \tilde{F}^{P, d^{\min}(P)} \geq \xi^{\min} \tilde{F}^{P, \bar{d}(P)} \geq \xi^{\min} Z_1^*$  (from Lemma 6.2). ■

### 6.3 Algorithms

#### 6.3.1 Delayed Production Algorithm for a Given Set of Prices

Given a price vector  $P$ , the algorithm below generates an optimal production and inventory policy for the Delayed Production model.

- *Step 1:* Solve the following dynamic program:

- *a: Initialization*

- \*  $J_{T+1} = 0$ ,  $G_T(0) = 0$ , and  $Y_t^* = S_t^* = 0$  for all  $t = 1, \dots, T$ ;
- \* For the last period, calculate the marginal expected revenue and salvage cost for each possible level of beginning available inventory, including production: For each  $Y$  from 1 to  $\sum_{t=1}^T q_t$ , calculate  $\Delta G_T(Y) = P_T[1 - \Psi_t^P(Y - 1)] + v\Psi_t^P(Y - 1)$  where  $v = \text{salvage value} - h_T < P_T$ .

- *b: Recursive equations*

Calculate  $J_t$ , the expected profit-to-go in period  $t$ :

- i.* First, find the optimal order-up-to level in period  $t$  by comparing production cost to the marginal expected revenue and inventory holding cost-to-go: Find  $Y_t^* = \max\{Y \text{ for } Y > 0 : k_t \leq \Delta G_t(Y)\}$ ;
- ii.* Calculate the expected revenue and inventory holding cost-to-go,  $G_t(Y)$ , from the marginal values:
  - \*  $G_t(0) = J_{t+1}(0)$ ;
  - \* For each  $Y$  from 1 to  $\sum_{t=1}^t q_t$ , calculate  $G_t(Y) = \Delta G_t(Y) + G_t(Y - 1)$ ;
- iii.* Finally, calculate the expected profit for each initial inventory level. The cases below correspond to: produce to maximum capacity, produce enough to raise the inventory level to the order-up-to value, or produce nothing if inventory

is larger than the order-up-to value. Calculate the marginal expected profit values as well.

For each  $I$  from 0 to  $\sum_{t=1}^{t-1} q_t$ ,

$$\text{calculate } J_t(I) = \begin{cases} -k_t q_t + G_t(I + q_t) & \text{if } I \leq Y_t^* - q_t \\ -k_t(Y_t^* - I) + G_t(Y_t^*) & \text{if } Y_t^* - q_t < I \leq Y_t^* \\ G_t(I) & \text{if } Y_t^* < I \end{cases}$$

For  $I > 0$ , calculate  $\Delta J_t(I) = J_t(I) - J_t(I - 1)$ .

Calculate  $\Delta G_{t-1}$ , the marginal expected revenue and inventory holding cost-to-go for period  $t - 1$ .

*i.* First, find the save-up-to target by comparing the price in period  $t - 1$  to the marginal expected profit-to-go minus inventory holding cost for one unit:

$$\text{Find } S_{t-1}^* = \max \{I \text{ for } I > 0 : P_{t-1} \leq \Delta J_t(I) - h_t\}$$

*ii.* Finally, calculate the marginal expected revenue and inventory holding cost-to-go from period  $t - 1$  forward, for each possible level of available inventory including production. The cases below are based on comparing the available inventory to the save-up-to target. If the available inventory is less, then everything is saved for the future. Otherwise, the amount of inventory for the future is conditioned on the realized demand in period  $t - 1$ .

For each  $Y$  from 1 to  $\sum_{t=1}^T q_t$ ,

$$\text{calculate } \Delta G_{t-1}(Y) = \begin{cases} \Delta J_t(Y) - h_t & \text{if } Y \leq S_{t-1}^* \\ \Delta J_t(Y) - h_t & \text{if } Y > S_{t-1}^* \\ P_{t-1} \left[ 1 - \Psi_{t-1}^P(Y - S_{t-1}^* - 1) \right] \\ \quad + \sum_{k < Y - S_{t-1}^*} [\Delta J_t(Y - k) - h_t] \psi_{t-1}^P(k). \end{cases}$$

– *c:* Increment  $t$  and return to recursive equations.

- *Step 2:* The decisions:

At the beginning of each period  $t$ ,

- Announce price  $P_t$ . (Alternatively, all prices may be announced at the beginning of the horizon.)
- Based on the on-hand inventory  $I_t$ , produce

$$X_t^* = \begin{cases} q_t & \text{if } I_t \leq Y_t^* - q_t \\ Y_t^* - I_t & \text{if } Y_t^* - q_t < I_t \leq Y_t^* \\ 0 & \text{if } Y_t^* < I_t. \end{cases}$$

and sell as much as possible after the first  $S_t^*$  units of inventory. That is, produce up to  $Y_t^*$  if possible, and sell all units in excess of  $S_t^*$ .

- Sales can be determined according to the following calculation:

$$D_t = \min(d_t(P), \max(X_t^* + I_t - S_t^*, 0)).$$

Theorem 2.1 thus implies that the algorithm finds the best order-up-to and save-up-to values in each period and that these two values are independent of the beginning inventory in each period.

### 6.3.2 Delayed Pricing Algorithm for a Given Production Schedule

Given a production vector  $X$ , the algorithm below generates an optimal pricing and inventory policy for the Delayed Pricing model.

- *Step 1:* Solve the following dynamic program:

- a: Initialization

- \*  $J_{T+1} = 0, G_T(0) = 0;$

- \* For the last period, calculate the marginal expected revenue and salvage cost for each possible level of beginning inventory, including production:

- For each  $Y$  from 1 to  $\sum_{t=1}^T X_t$ ,
    - calculate  $\Delta G_T(Y) = \max_{P_T} P_T [1 - \Psi_t^{P_t}(Y - 1)] + v \Psi_t^{P_t}(Y - 1)$

- where  $v = \text{salvage value} - h_T < P_T$ .

- \* Let  $P_T^*(Y)$  be the price that maximizes  $\Delta G_T(Y)$ .

- b: *Recursive equations*

Calculate  $J_t$ , the expected profit-to-go in period  $t$ .

- i. Calculate the expected revenue and inventory holding cost-to-go,  $G_t(Y)$ , from the marginal values.

- \*  $G_t(0) = J_{t+1}(0);$

- \* For each  $Y$  from 1 to  $\sum_{t=1}^t X_t$ ,
  - calculate  $G_t(Y) = \Delta G_t(Y) + G_t(Y - 1)$ .

- ii. Finally, calculate the expected profit and marginal expected profit for each initial inventory level, using the production schedule determined at time 0.

- For each  $I$  from 0 to  $\sum_{t=1}^{t-1} X_t$ ,
  - calculate  $J_t(I) = -k_t(X_t) + G_t(I + X_t)$ .

- For  $I > 0$ , calculate  $\Delta J_t(I) = J_t(I) - J_t(I - 1)$ .

Calculate  $\Delta G_{t-1}$ , the marginal expected revenue and inventory holding cost-to-go for period  $t - 1$ .

- i. Calculate  $\Delta G_{t-1}$  for each possible beginning level of available inventory, including production. For each beginning inventory level, the calculation searches for the best price and best save-up-to target. The cases below are based on comparing the available inventory to the save-up-to target. If the available inventory is less, then everything is saved for the future. Otherwise, the amount of inventory for the future is conditioned on the realized demand in period  $t - 1$ .

For each  $Y$  from 1 to  $\sum_{t=1}^t X_t$ , calculate

$$\Delta G_{t-1}(Y) = \max_{P_{t-1}} \left\{ \begin{array}{l} P_{t-1} [1 - \Psi_{t-1}^{P_{t-1}}(Y - 1)] \\ + \sum_{k < Y} [\Delta J_t(Y - k) - h_t] \psi_{t-1}^{P_{t-1}}(k). \end{array} \right.$$

ii. Let  $P_{t-1}^*(Y)$  be the price that maximizes  $\Delta G_t(Y)$

– *c*: Increment  $t$  and return to recursive equations.

- *Step 2*: The decisions:

At the beginning of each period  $t$ ,

- a. Produce  $X_t$  units of product.
- b. Based on the on-hand inventory  $I_t$ , announce price  $P_t^*(I_t + X_t)$ , and sell as much as possible after the first  $S_t^*(I_t + X_t)$  units of inventory.

The optimal save-up-to value,  $S_t^*$ , and the optimal price  $P_t^*$  are now functions of period  $t$  initial inventory level, and thus the optimal policy, unlike the Delayed Production case, is not a threshold type.