A Portfolio Approach to Procurement Contracts

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Abstract

The purpose of this paper is to develop a general framework for supply contracts in which portfolios of contracts can be analyzed and optimized. We focus on a multi-period environment with convex supply contracts, convex spot market costs and convex inventory holding costs. We characterize the optimal replenishment policy for a portfolio consisting of long-term and option contracts and show that this policy has a simple structure. Specifically, we show that every source of supply is managed using a modified base-stock policy. We also provide structural properties that allow to efficiently design an optimal portfolio. In particular, we derive conditions to identify dominated contracts, and show the relationship between the amount of capacity reserved and the base-stock levels. Finally, we present computational results, that illustrate the sensitivity of the optimal portfolio to several of the parameters involved in the model. Our experiments also indicate that portfolio contracts not only increase the manufacturer's expected profit but can also reduce its financial risk.

1 Introduction

A recent trend for many industrial manufacturers has been outsourcing; firms are considering outsourcing everything from production and manufacturing to the procurement function itself. Indeed, in the mid 90s, there was a significant increase in purchasing volume as a percentage of the firm's total sales. More recently, between 1998 and 2000, outsourcing in the electronics industry has increased from 15 percent of all components to 40 percent, as noted in [15].

For instance, throughout the 90s, strategic outsourcing, i.e., outsourcing the manufacturing of key components, was used as a tool to rapidly cut costs. In their recent study, Lakenan, Boyd and Frey [12] reviewed the case of eight major Contract Equipment Manufacturers (CEMs) – Solectron, Flextronics, SCI Systems, Jabil Circuit, Celestica, ACT Manufacturing, Plexus and Sanmina – which were the main suppliers to Original Equipment Manufacturers (OEMs) such as Dell, Marconi, NEC Computers, Nortel, and Silicon Graphics. The aggregated revenue for the eight CEMs quadrupled between 1996 and 2000 while their capital expenditure grew 11-fold.

Of course, the increase in the level of outsourcing implies that the procurement function becomes critical for a manufacturer to remain in control of its destiny. As a result, many OEMs focus on closely collaborating with the suppliers of their *strategic components*. In some cases, this is done using *private exchange* market-places and/or effective supply contracts; both of these try to coordinate the supply chain.

A different approach has been applied for *non-strategic components*. In this case, products can be purchased from a variety of suppliers and flexibility to market conditions is perceived as more important than a permanent relationship with the suppliers. Indeed, *commodity products*, e.g., electricity, steel, grain, cotton or computer memory, are typically available from a large number of suppliers and can be purchased in spot markets. Because these are highly standard products, switching from one supplier to another is not considered a major problem.

Despite the non-strategic nature of commodity products, it is critical to identify effective procurement strategies for these components, since manufacturers may be completely dependent on

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them. For instance, production costs might be very sensitive to the cost of some commodity products, e.g., electricity for automobile manufacturers, or memory for computer manufacturers. At the same time, uncertainty in supply and customer demand raises the question of whether to purchase supply now or wait for better market conditions in the future.

Thus, an effective procurement strategy for commodity products has to focus on both driving costs down and reducing risks. These risks include both *inventory* and *price* risks. By inventory risk we refer to inventory shortages or unsold products while price risk refers to the purchasing price which is uncertain if the procurement strategy depends on spot markets.

A traditional procurement strategy that eliminates price risk is the use of *long-term* contracts, also called *forward buy* or *fixed commitment* contracts. These contracts specify a fixed amount of supply to be delivered at some point in the future; the supplier and the manufacturer agree on both the price and the quantity delivered to the manufacturer. Thus, in this case, the manufacturer bears no price risk while taking huge inventory risk due to uncertainty in demand and the inability to adjust order quantities.

One way to reduce inventory risk is through *option* contracts, in which the buyer pre-pays a relatively small fraction of the product price up-front, in return for a commitment from the supplier to reserve capacity up to a certain level. The initial payment is typically referred to as *reservation price* or *premium.* If the buyer does not exercise the option, the initial payment is lost. The buyer can purchase any amount of supply up to the option level, by paying an additional price, agreed to at the time the contract is signed, for each unit purchased. This additional price is referred to as *execution price* or *exercise price.* Of course, the total price (reservation plus execution price) paid by the manufacturer for each purchased unit is typically higher than the price of a long-term contract.

Option contracts provide the manufacturer with flexibility to adjust order quantities depending on realized demand and hence these contracts reduce inventory risk. Thus, these contracts shift risk from the manufacturer to the supplier since the supplier is now exposed to customer demand uncertainty. This is in contrast to long-term contracts in which the manufacturer takes all the risk.

A different strategy used in practice to share risk between suppliers and manufacturers is through *flexibility* contracts. In these contracts, a fixed amount of supply is determined when the contract is signed, but the amount to be delivered and paid for can differ by no more than a given percentage determined upon signing the contract.

Interestingly, as we show later on, flexibility contracts are equivalent to a combination of a long-term contract plus an option contract. This observation suggests that a procurement strategy that combines several contracts at the same time can be beneficial to both the manufacturer and the supplier.

The objective of the current paper is to develop a framework that provides *manufacturers* with the ability to select multiple contracts at the same time in order to optimize their expected profit. This approach is particularly meaningful for commodity products since a large pool of suppliers is available, each offering a different type of contract. Thus, the manufacturer may be interested in selecting several different complementary contracts so as to reduce expected procurement and inventory holding costs. As we shall illustrate, this strategy can also drive profit uncertainty down while simultaneously increasing expected profit.

For this purpose, we define a new type of contract, called *portfolio* contract, that is in fact a combination of many traditional contracts, such as long-term contracts, options and flexibility contracts. Of course, in the design of such a portfolio contract one may take into account the spot market since this is also a source of supply for commodity products.

Interestingly, current trends in industry are also directed toward the analysis and optimization of supply contracts. Most relevant to this paper is the work done at Hewlett-Packard (HP) on the use of a portfolio approach for the procurement of electricity or memory products, see Billington [3]. Specifically, 50% of HP's procurement cost is invested in long-term contracts, 35% in option contracts and the remaining is left to the spot market, see [9].

The academic literature on supply contracts is quite recent. For a review see Cachon [7] or Lariviere [13]. As observed in [13], the literature can be classified into two main categories. The first focuses on replenishment policies and detailed contract parameters for a given type of contract. Examples include Anupindi and Bassok [1] for flexibility contracts, Brown and Lee [5] for option contracts applied in the semiconductor industry setting, or Wu, Kleindorfer and Zhang [17], Kleinknecht and Akella [11], Spinler, Huchzermeier and Kleindorfer [16] or Golovachkina and Bradley [10] for option contracts in the presence of a spot market. Typically, the objective in this category is to optimize the buyer's procurement strategy with very little regard to the impact of the decision on the seller. The second category focuses on optimizing the terms of the contract so as to improve supply chain coordination. Examples here include buy-back contracts, see Pasternack [14], revenue sharing contracts, see Cachon and Lariviere [8], or option contracts, see Barnes-Schuster, Bassok and Anupindi [2]. Unlike the first category, here the objective is to characterize contracts that allow each party to optimize its own profit but lead to a globally optimized supply chain.

The current paper belongs to the first category. Specifically, we analyze the general case of a portfolio of contracts taking into account the presence of the spot market. Our objective is to design effective portfolio contracts and identify the replenishment policy so as to maximize the manufacturer's (i.e., the buyer) expected profit.

For this purpose, we start in Section 2 by describing a general finite horizon model and formally introduce our assumptions. In Section 3, we formulate the dynamic program required to identify both the optimal portfolio contract and the replenishment strategy and we analyze the properties of the optimal replenishment strategy for general contracts. We then specify these results for portfolios of option contracts. In Section 4, we characterize the structure of optimal procurement strategies and in Section 5 we deal with the design of optimal portfolios. We show in Section 6 our numerical experiments where we analyze the sensitivity of the optimal portfolios to different parameters and we compute estimated distributions of profits for different contracts. Finally, in Section 7 we present several extensions of this model and in Section 8 we discuss future directions of research.

2 Model Formulation

Consider a manufacturer of a single item who faces demand during several time periods. Future demands are unknown but as time goes by, more precise information on the distribution of demand becomes available. The manufacturer's objective is to optimally manage its supply by buying well-chosen capacity for every time period. Specifically, the objective is to maximize profit by effectively managing the supply process.

For this purpose, the manufacturer needs the right trade-off between price and flexibility. That is, the manufacturer needs to find the appropriate mix of low price yet low flexibility (i.e., long-term) contracts, reasonable price but better flexibility (i.e., option) contracts or unknown price and quantity supply but no commitment (i.e., the spot market). Once these decisions are made at the beginning of the horizon, the manufacturer has to manage its inventory effectively, which can be viewed as an extra supply source, carried over from period to period.

2.1 Sequence of events

At the beginning of the planning horizon, i.e. in period $t = 0$, the manufacturer decides on the type of contact it will buy from its suppliers for the entire planning horizon. This planning horizon is finite and has T time periods indexed from 1 to T in an increasing order. The contract specifies for every time period the cost of receiving any amount of supply; see Section 2.3 for a detailed definition.

At the beginning of period t, $t = 1, \ldots, T$, the customer demand d_t and the spot market price $s_t(\cdot)$ become known. At that time, the manufacturer decides how much of the contract to execute. Of course, the manufacturer is also able to buy supply directly from the spot market. In any event, the inventory carried from previous periods plus the incoming supply can be used to satisfy this period's demand. The remaining inventory is carried over to the next period. Finally, unsatisfied demand is lost to the competition.

2.2 Example

Consider a manufacturer that needs to find supply sources for electricity. The manufacturer produces and sells products at a unit price, $p = 20$, and we assume that the only contributor to the production cost is the cost of electricity. One way to interpret p is as the profit margin, i.e. the final price of the product minus the loaded costs from other components.

To simplify the example we assume that a unit of electricity is required to produce a unit of finished good. The manufacturer thus has information on the distribution of the potential electricity demand. More precisely, it knows that demand for electricity follows a truncated normal distribution of mean $\mu = 1000$ and standard deviation $\sigma = 300$.

Three power companies are available for supply:

- Company 1 offers a long-term contract with the following conditions: power is bought in advance at a price $v_1 = 10$ per unit, and there is no price to pay at delivery ($w_1 = 0$).
- Company 2 offers an option contract with payment of $v_2 = 6$ per unit paid in advance and then $w_2 = 6$ per unit paid upon delivery.
- Company 3 has similar terms to company 2, but with prices $v_3 = 3$ per unit paid in advance and $w_3 = 12$ per unit paid upon delivery.

The manufacturer problem is to find the right balance between the different contracts: how much to commit from the long-term contract? how much capacity to buy for each one of the two option contracts? and, how much supply to leave uncommitted?

Assume now that there also exists a spot market for electricity, such that the spot market price is uncorrelated with the specific demand for electricity generated by the manufacturer. Supply from this spot market is unlimited and must be paid at a unit price S, where S follows a uniform distribution in $[10, 20]$.

How does the optimal solution to the previous problem change when it is possible to use the spot market as an extra source? We solve this problem in Section 6.1.

2.3 Supply contract

The contract purchased at time period $t = 0$ will be denoted by $\mathbf{r}(\cdot) = (\ r_1(\cdot), \ldots, r_T(\cdot)\ )$. It is a vector of functions that should be interpreted as follows: at time period t, the contract allows the manufacturer to buy an amount $q_t \geq 0$ at a total cost of $r_t(q_t)$.

Assumption 1 *For $t = 1, \ldots, T$, the function $r_t(\cdot)$ is convex.*

To justify the assumption observe that when the manufacturer faces customer demand, it must choose among the different sources included in the contract. Evidently, to minimize costs, the manufacturer will prefer the cheapest unit-price sources. Thus, given that it executes an amount q, the incurred cost will be convex in q.

In some industrial settings, the manufacturer may be offered volume discounts. These discounts typically take the form of a discount on the reservation price and hence are counted as a fixed payment. Once this contract has been determined, the variable payment (as a function of the executed quantity) is typically convex. Therefore, Assumption 1 is often satisfied in practice, since the discounts are counted as a fixed component of $r_t(\cdot)$. The next example illustrates that Assumption 1 holds even under volume discount on the reservation price.

Example 1 *Consider a portfolio contract made up of n options. Each option has a maximum capacity x_i, a pre-paid reservation cost (paid at the beginning of the horizon) $v_i(x_i)$ and an execution cost $w_i \geq 0$ per unit delivered (paid when executing part or all of this option), for each $i = 1, \ldots, n$. Given that we must buy an amount q of supply, we have*

$$r(q) = \sum_{i=1}^{n} v_i(x_i) + \quad \min \sum_{i=1}^{n} w_i q_i \quad \text{subject to} \quad \begin{cases} \sum_{i=1}^{n} q_i = q \\ 0 \leq q_i \leq x_i \ \forall i = 1, \ldots, n. \end{cases}$$

This linear program is clearly be convex in q.

Note that a special case of this portfolio contract is one in which long-term commitment is included as one option in the portfolio. That is, in this case, $(v_1(\cdot) \geq 0, w_1 = 0, x_1)$ are the parameters of the long-term portion of the portfolio. This implies that at the beginning of the horizon the manufacturer pays $v_1(x_1)$ and has access to supply up to x_1 units at no additional cost. The portfolio may have additional options, $(v_2(\cdot), w_2, x_2), \ldots, (v_n(\cdot), w_n, x_n)$. For each option i, $i = 2, \ldots, n$, the manufacturer pre-pays at the beginning of the horizon an amount $v_i(x_i)$, and as a result, it is able to execute no more than x_i units at a price of w_i per unit.

Thus, this modeling approach is quite general and has the advantage that it is not restricted to a fixed class of contracts such as long-term contracts, flexibility contracts or option contracts. Figure 1 provides an example of the shape of the function $r(\cdot)$.

2.4 Spot market

Since our focus in this paper is on commodity products, we assume that there is a supply market that can be used by the manufacturer to purchase, at any time period, additional components. Thus, at period t, the manufacturer obtains an amount $q_t \geq 0$ of supply at a total cost $s_t(q_t)$. This spot market cost function, $s_t(\cdot)$, is random and will only be known at the beginning of period t. Prior to that time, the manufacturer has only probabilistic information on the spot market costs $s_t(\cdot)$. Because our model can handle a learning process, the distribution of the spot market cost can be improved as time goes by. More on this is discussed in the next section.

Assumption 2 *For $t = 1, \ldots, T$, the random function $s_t(\cdot)$ is a convex function for all outcomes.*

This assumption is equivalent to saying that in the spot market, the marginal unit cost is increasing with quantity. This is a natural assumption since the spot market has limited supply with many competing buyers and hence the more the manufacturer purchases from the spot market the higher the marginal price it has to pay.

In some industries, as mentioned in the previous section, manufacturers may face volume discounts. We argue that these are not relevant for the spot market modeling. Indeed, spot market purchases are typically made to complement shortages in long-term planning, and this implies small

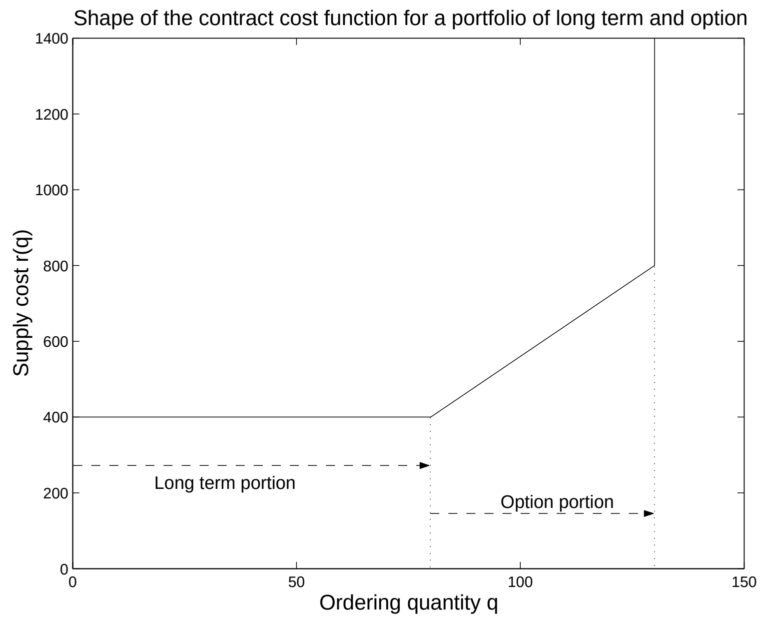

Figure 1: Example of portfolio contract: combination of long-term contract and option contract.

volumes and fast delivery. On the other hand, volume discounts are available when the orders are made long in advance. In this case, the volumes are important and the production time is long so that economies of scale can be created through better planning and scheduling.

A special case captured by the model is the case in which the spot market unit price at time t is constant, equal to S_t, and the market has limited capacity κ_t. The vector (S_t, κ_t) is a random variable revealed only in period t. In particular, if there is no spot market, we can select either $S_t = \infty$ or $\kappa_t = 0$.

Note that in the current model the manufacturer is not allowed to sell inventory back to the spot market since the function is defined for $q_t \geq 0$. This is indeed the case for engineered products, that are tailored for the manufacturer, such as cell phone displays. However, as we observe in the extensions, all our results can be extended to situations in which the manufacturer can sell inventory back to the spot market, as long as the function $s_t(\cdot)$ remains convex.

2.5 Definition of the information state

At every time period, t, $t = 0, \ldots, T-1$, the manufacturer knows the probability distribution of future events such as demand $(D_k)_{k=t+1,\ldots,T}$ and spot market cost structure $(s_k(\cdot))_{k=t+1,\ldots,T}$.

We denote by Φ_t the information state at time t and we define Φ recursively in the following way:

$$\Phi_t = (\Phi_{t-1}, D_t, s_t(\cdot)) \; \forall t = 1, \ldots, T.$$

Assumption 3 *The distribution of future customer demand and spot market cost structure depends only on Φ_t, not on past decisions made by the manufacturer.*

Thus, we assume that future events are independent of the decisions that the manufacturer took in the past. Note that future events do not depend on decisions made by the firm so far but may depend on past demand or past spot market cost structures. In other words, we assume that the manufacturer is small relative to the market and its behavior cannot affect future demand or market prices. Since we make no assumption on the relationship between demand and spot market cost structure, we allow the spot market prices to be correlated with customer demands.

Since the distributions of future demand and spot market price depend only on Φ_t, and Φ_t is determined by past events, this modeling approach allows the manufacturer to improve its forecast as more information is available. For instance, as the manufacturer approaches period t it may be the case that the standard deviation of period t demand can be reduced. This is the *learning process* we refer to in this paper.

2.6 Modeling inventory

We denote by $\mathbf{I} \in \mathbb{R}^{T+1}$ the vector of inventory levels. I_t, for $t = 1, \ldots, T+1$, is the inventory level at the beginning of period t. To simplify the presentation, we assume $I_1 = 0$, although the model can handle any initial inventory level.

Since this is a lost sales model, we impose $\mathbf{I} \geq 0$. This implies that the manufacturer may lose some of the demand, however this loss will not affect future demand. Note that the inventory position is known at all times. Demand is known before stock is replenished, so the system moves from a known inventory position to another known inventory position after adding replenishment and subtracting demand. We can thus make sure that inventory levels remain non-negative.

As is common in traditional inventory models, an inventory holding cost $h_t(I_t)$ is incurred at the beginning of period t. The family of functions $h_t(\cdot)$, $t = 2, \ldots, T$, is known in advance, as it is contained in Φ_0.

Assumption 4 *For $t = 2, \ldots, T$, the function $h_t(\cdot)$ is convex and non-decreasing.*

This is a standard assumption used in periodic review inventory models. It is typically satisfied in practice, since the per-unit cost of keeping inventory increases due to scarcity of resources.

2.7 Prices and salvage value are exogenous

Every period, the manufacturer receives the amount ordered at that period and sells products to the end customers at a price p_t.

Assumption 5 *The vector of prices $\mathbf{p} \in \mathbb{R}_+^T$ at which the manufacturer charges the end customers is decided exogenously, in advance.*

This assumption implies that the pricing decision is not taken jointly with contract negotiation and inventory replenishment decisions. Even though joint optimization could yield important improvement, in most cases companies manage these two decisions separately, pricing through marketing or sales divisions, and purchasing and inventory management through procurement/purchasing and operations divisions. It is then realistic to take prices as a given input in the problem.

In addition, we assume that at the end of the planning horizon, remaining inventory is sold at salvage unit price $a > 0$. This implies that the manufacturer will receive a revenue of aI_{T+1} if I_{T+1} items are left at the end of the horizon. This salvage value is pre-specified in advance as well.

Assumption 6 *The salvage value $a \in \mathbb{R}_+$ that the manufacturer may obtain for unused inventory at the end of the horizon is determined exogenously, in advance.*

For technical reasons, namely to make sure that expected profit is finite, we need the following assumption.

Assumption 7 *For $t = 1, \dots, T$, for q big enough, $\forall \mu \in \partial r_t(q)$ and $\forall \nu \in \partial s_t(q)$ (in the subgradients of r_t and s_t) $\mu > a$ and $\nu > a$.*

The assumption simply says that it is not possible for the manufacturer to buy large quantities, either through the contract or the spot market, and sell for profit at the end of the planning horizon.

3 Replenishment Strategies

The decisions the manufacturer must take can be analyzed using dynamic programming, formulated from the last period to the first one, i.e., from $T+1$ to 0.

At each time period $t = 1, \dots, T+1$, the state space is defined by

- the inventory position I_t;
- the information vector Φ_t, which contains the current demand to be served, d_t, and the spot market description $s_t(\cdot)$;
- the supply contract $\mathbf{r}(\cdot)$.

At time period $t = 0$, we only have the information vector Φ_0, given a priori. At time period $t = T+1$, the only relevant information is the remaining inventory, I_{T+1}.

For every period t, $t = 1, \dots, T$, the decision space includes the following quantities:

- the quantity $q_t^r \geq 0$ that the manufacturer executes from the supply contract at a cost $r_t(q_t^r)$;
- the quantity $q_t^s \geq 0$ that the manufacturer purchases from the spot market at a cost $s_t(q_t^s)$;
- the amount of demand not satisfied q_t^n, $0 \leq q_t^n \leq d_t$.

These decisions completely characterize the inventory position at the beginning of period $t+1$, I_{t+1}, where $I_{t+1} = I_t + q_t^r + q_t^s + q_t^n - d_t$. Since this is a lost sales model, we must have $I_{t+1} \geq 0$.

At the end of the horizon, the manufacturer sells the remaining inventory at salvage value, a. Therefore, the profit-to-go function at this period is

$$V_{T+1}(I_{T+1}, \Phi_{T+1}, \mathbf{r}(\cdot)) = aI_{T+1}.$$

For every t, $t = 1, \dots, T$, the profit-to-go function at period t can be written as

$$V_t(I_t, \Phi_t, \mathbf{r}(\cdot)) = \max_{q_t^r, q_t^s, I_{t+1}} \left\{ \begin{array}{l} p_t(I_t - I_{t+1} + q_t^r + q_t^s) \\ -r_t(q_t^r) - s_t(q_t^s) \\ -h_t(I_t) \\ +\mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1}, \mathbf{r}(\cdot)) \end{array} \right\} \text{ subject to } \left\{ \begin{array}{l} q_t^r \geq 0 \\ q_t^s \geq 0 \\ I_{t+1} \geq 0 \\ I_t - I_{t+1} + q_t^r + q_t^s \geq 0 \\ I_t - I_{t+1} + q_t^r + q_t^s \leq d_t \end{array} \right. \quad (1)$$

Observe that in this formulation the decision variables are q_t^r, q_t^s and I_{t+1} which is equivalent to optimizing on the variables q_t^r, q_t^s and q_t^n.

Finally, in period $t = 0$, the manufacturer has to choose one contract $\mathbf{r}(\cdot)$ among a family of contracts $\mathcal{R}$. The manufacturer's contract selection problem is thus,

$$V_0(\Phi_0) = \max_{\mathbf{r}(\cdot) \in \mathcal{R}} \mathbb{E}_{\Phi_1} V_1(I_1 = 0, \Phi_1, \mathbf{r}(\cdot)).$$

We are now ready to analyze the optimal inventory replenishment policy given the current information and a choice of $\mathbf{r}(\cdot)$.

3.1 Preliminaries

The following lemma will be applied in various parts of the analysis.

Lemma 1 *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to R$ be a concave function. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Define $P(b) = \{x \in \mathbb{R}^n | Ax \le b\}$ and*

$$g(b) = \begin{array}{l} \max f(x,b) \\ s.t.\ x \in P(b) \end{array} \tag{2}$$

Let $Q = \{b \in \mathbb{R}^m | P(b) \neq \emptyset\}$. Then Q is a convex set and $g : Q \to R$ is a concave function.

All the proofs are presented in the Appendix.

3.2 Optimal policy

In this part, we drop $\mathbf{r}(\cdot)$ from the notation because it is fixed through all the time periods $t = 1, \dots, T+1$.

Proposition 1 *Consider any time period t, $t = 1, \dots, T+1$. Given Φ_t, the profit-to-go function $V_t(I_t, \Phi_t)$ is concave in I_t.*

The proposition thus implies that the marginal profit from every additional unit of inventory is non-increasing with the inventory level.

We now characterize the optimal replenishment policy. Given time period t and Φ_t, define function U_{t+1} as follows.

$$U_{t+1}(I_{t+1}) = \mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1}). \tag{3}$$

By applying Proposition 1 and taking the expectation on Φ_{t+1}, we see that U_{t+1} is concave in I_{t+1}.

Observe that the optimization problem defined by Equation (1) can be rewritten as

$$V_t(I_t, \Phi_t) = p_t d_t - h_t(I_t) + \max_{I_{t+1}} -C_t(I_{t+1} - I_t + d_t) + U_{t+1}(I_{t+1}) \quad \text{subject to} \begin{cases} I_{t+1} \ge 0 \\ I_{t+1} \ge I_t - d_t, \end{cases} \tag{4}$$

where the function C_t is defined as follows.

$$C_t(z_t) = \min_{q_t^r, q_t^s, q_t^n} p_t q_t^n + r_t(q_t^r) + s_t(q_t^s) \quad \text{subject to} \begin{cases} q_t^r \ge 0 \\ q_t^s \ge 0 \\ q_t^n \ge 0 \\ q_t^n \le d_t \\ q_t^r + q_t^s + q_t^n = z_t. \end{cases} \tag{5}$$

Lemma 1 shows that C_t is convex in z_t. Intuitively, this is explained as follows. The optimal policy that produces a total of z_t units starts producing using cheaper means first and applies more expensive means later.

This property is used in the proof of the next proposition. This proposition states that the optimal inventory target level as a function of the on-hand inventory is increasing and non-expanding. In other words, the higher the inventory level at the beginning of a given period, I_t, the higher the inventory level at the end of the period, I_{t+1}; similarly, the higher the inventory level, I_t, the smaller the ordering quantity, $z_t = I_{t+1} - I_t + d_t$.

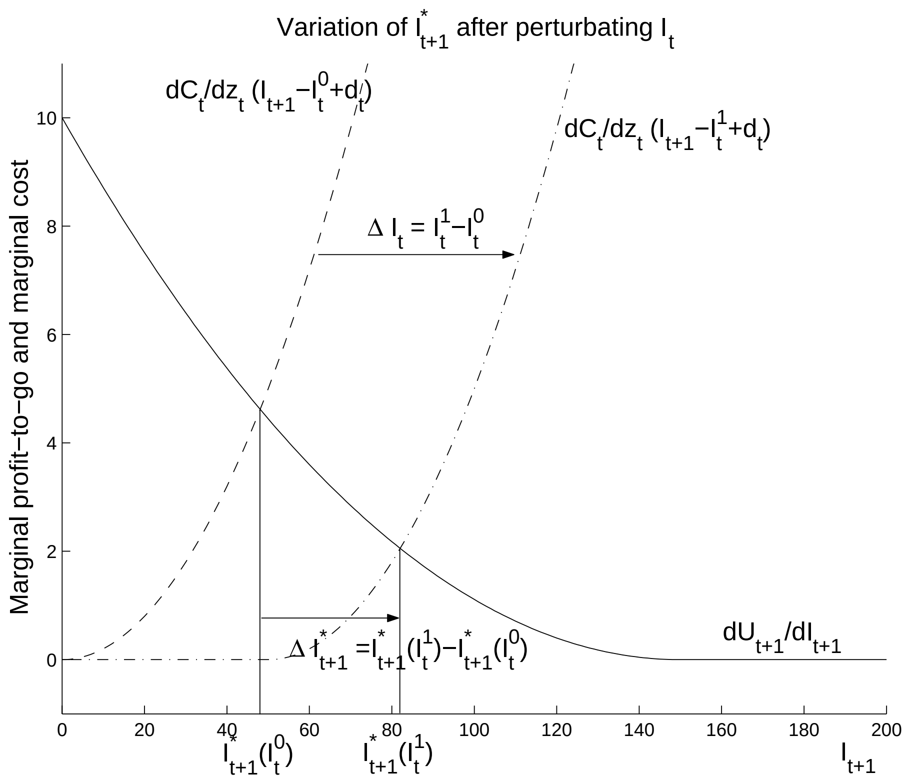

Figure 2: Illustration of Proposition 2.

Proposition 2 *Given Φ_t, $\forall t = 1, \ldots, T$, define $I^*_{t+1}(I_t)$ to be the smallest optimal control for the next period inventory level in the optimization problem defined by Equation (4) with parameter I_t. Then, for every $I^1_t \geq I^0_t \geq 0$, we have*

$$0 \leq I^*_{t+1}(I^1_t) - I^*_{t+1}(I^0_t) \leq I^1_t - I^0_t$$

A sketch of the proof of the proposition, in the differentiable case, uses Figure 2. In this case, the optimality condition is

$$\frac{dU_{t+1}}{dI_{t+1}}_{|I^*_{t+1}} = \frac{dC_t}{dz_t}_{|z^*_t}$$

subject to the constraint that $I_{t+1} = I_t - d_t + z_t$. To determine the optimal I_{t+1}, it is sufficient to find the intersection of the non-increasing function $\frac{dU_{t+1}}{dI_{t+1}}$ and the non-decreasing function $I_{t+1} \to \frac{dC_t}{dz_t}(-I_t + d_t + I_{t+1})$. Therefore, the parameter I_t is used to determine by how much we shift the graph of $\frac{dC_t}{dz_t}$. When we increase I_t from I^0_t to I^1_t, we shift the graph of $z \to \frac{dC_t}{dz_t}(-I_t + d_t + I_{t+1})$ to the right by $I^1_t - I^0_t$, and hence, since $\frac{dU_{t+1}}{dI_{t+1}}$ is non-increasing, the optimal inventory position at the next time period I_{t+1} cannot decrease. Moreover, since we have shifted the graph of $\frac{dC_t}{dz_t}$ by $I^1_t - I^0_t$, the intersection cannot happen after $I^0_{t+1} + I^1_t - I^0_t$.

In fact, in the differentiable case, the proposition below characterizes the behavior of I^*_{t+1} as a function of I_t.

Proposition 3 *Given Φ_t, assume that the functions U_{t+1} and C_t are twice differentiable at $I^*_{t+1}(I_t)$ and $I^*_{t+1}(I_t) - I_t + d_t$ respectively. Then if*

$$\frac{d^2C_t}{dz_t^2}_{|I^*_{t+1}(I_t)-I_t+d_t} - \frac{d^2U_{t+1}}{dI_{t+1}^2}_{|I^*_{t+1}(I_t)} \neq 0$$

and I^*_{t+1} is differentiable at I_t we have that

$$\frac{dI^*_{t+1}}{dI_t}_{|I_t} = \frac{\frac{d^2C_t}{dz_t^2}_{|I^*_{t+1}(I_t)-I_t+d_t}}{\frac{d^2C_t}{dz_t^2}_{|I^*_{t+1}(I_t)-I_t+d_t} - \frac{d^2U_{t+1}}{dI_{t+1}^2}_{|I^*_{t+1}(I_t)}}.$$

This proposition can be very powerful in practice, since it may reduce computational effort. If, for a given I_t^0, we know the optimal next-period level $I_{t+1}^0 = I^*_{t+1}(I_t^0)$, we can use this formula to generate the optimal control for any other I_t by generating the optimal controls $I^*_{t+1}(I)$ for $I \in [I_t^0, I_t]$ (or $[I_t, I_t^0]$ when $I_t < I_t^0$). In fact, in a prototype implementation of the results reported here, this property allowed us to reduce running time by a significant factor.

4 Replenishment Policies for Portfolio Contracts

So far we have focused on general, convex contracts; i.e., for every time period, total purchasing cost was convex in the amount purchased. We now focus on the special case of portfolio contracts which include option contracts only. Of course, long-term commitments may be included in this framework, as described immediately after Example 1.

This is the case, for instance, when a manufacturer receives offers from different suppliers offering different option terms, or when a single supplier offers a broad spectrum of option contracts. As we will see in Example 3, this formulation captures a wide class of contracts, including flexibility contracts. Again, our focus in this section is on the optimal replenishment strategy for a given portfolio.

4.1 Assumptions and notation

Assumption 8 *The spot market unit cost is constant and equal to S_t, and the market only offers a limited supply κ_t, for $t = 1, \ldots, T$. That is, the structure of $s_t(q)$ is as follows.*

$$s_t(q) = \begin{cases} S_t q & \text{for } 0 \leq q \leq \kappa_t \\ +\infty & \text{else} \end{cases}$$

Of course, the spot market unit price and capacity are random variables realized at the beginning of each time period. The previous modeling assumption is based on our industry experience where we observe manufacturers using these two variables to describe the spot market.

The contract is defined as a portfolio of simple options. We do not consider yet the up-front cost of buying the options because it is a fixed cost, that has no impact on the replenishment policy applied in period $t > 0$.

Assumption 9 *For $t = 1, \ldots, T$, the contract $r_t(\cdot)$ is made up of n_t options with capacities $x_t^{i_t}$ and execution unit price $w_t^{i_t}$, $i_t = 1, \ldots, n_t$. Without loss of generality we assume that $w_t^1 \leq \ldots \leq w_t^{n_t}$.*

Therefore, the total execution cost associated with these options is determined by solving,

$$r_t(q) = \min \sum_{i=1}^{n_t} w_t^i q_t^i \quad \textit{subject to} \quad \begin{cases} \sum_{i=1}^{n_t} q_t^i = q \\ 0 \le q_t^i \le x_t^i \ \forall i = 1, \ldots, n_t. \end{cases}$$

With these assumptions, the manufacturer faces $n_t + 2$ supply sources, which are:

- n_t simple options of unit cost w_t^i and capacity x_t^i for $i = 1, \ldots, n_t$;
- the spot market, which offers supply at unit cost S_t up to a capacity of κ_t;
- not to serve demand; the firm can choose not to fulfill demand, which is equivalent to serving all demand by buying supply at a unit cost p_t up to a capacity of D_t.

Each of these sources offers supply at fixed marginal cost (execution price) for a given capacity. These supply sources can be represented as a pair (unit cost, capacity level): (w_t^i, x_t^i) for $i = 1, \ldots, n_t$, (S_t, κ_t) and (p_t, D_t). Define $\overline{n}_t = n_t + 2$ and sort these pairs by increasing unit cost. Let $(\overline{w}_t^j, \overline{x}_t^j)$ be the pair with j^{th} smallest unit cost. Thus, $\overline{w}_t^1 \le \ldots \le \overline{w}_t^{\overline{n}_t}$.

4.2 Optimal replenishment policy

The new definition of the supply sources, $(\overline{w}_t^j, \overline{x}_t^j)$, allows us to rewrite Equation (5) as follows,

$$C_t(z_t) = \min \sum_{i=1}^{\overline{n}_t} \overline{w}_t^i q_t^i \quad \text{subject to} \quad \begin{cases} \sum_{i=1}^{\overline{n}_t} q_t^i = z_t \\ 0 \le q_t^i \le \overline{x}_t^i \ \forall i = 1, \ldots, \overline{n}_t \end{cases} \tag{6}$$

Theorem 1 *Given Φ_t, for $t = 1, \ldots, T$, there exist inventory target levels $b_t^1, \ldots, b_t^{\overline{n}_t}$ such that*

(i) for $i = 1, \ldots, \overline{n}_t - 1$, $b_t^i \le b_t^{i+1} - \overline{x}_t^{i+1}$;

(ii) for $i = 1, \ldots, \overline{n}_t$, it is optimal to order from source i

- 0 *if* $I_t \ge b_t^i$;
- $b_t^i - I_t$ *if* $b_t^i - \overline{x}_t^i \le I_t \le b_t^i$;
- $\overline{x}_t^i$ *otherwise.*

The theorem thus completely characterizes the optimal replenishment policy for a portfolio of option contracts. In particular, the optimal policy is obtained by *following a modified base-stock policy for every option, the spot market, and the demand source.* The ordering of the contracts implies that contracts are executed based on execution price, starting with the contract with the cheapest price. Of course, a contract with execution price higher than the spot price will only be used once the manufacturer has exhausted the spot market capacity.

The aggregated policy is shown in Figure 3. We see that the functions $I_{t+1}(\cdot)$ and $z_t(\cdot)$ have slope 0 or 1, or -1 or 0 respectively. This implies that for a given interval of I_t, i.e. the initial inventory level at the beginning of period t, either the order quantity does not change or the target inventory level remains constant.

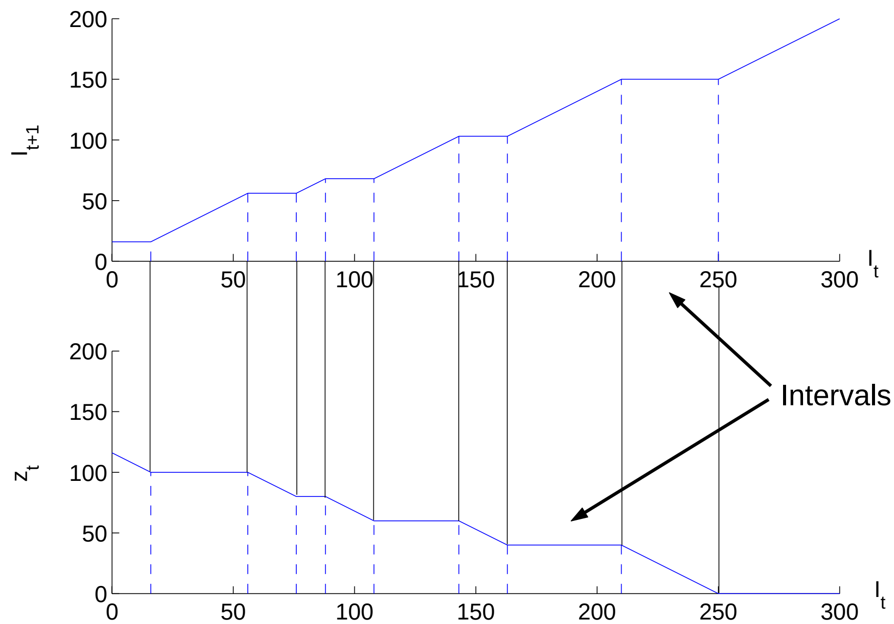

Figure 3: Instance of the optimal replenishment policy in terms of $I^*_{t+1}(I_t)$ and $z^*_t(I_t)$.

5 Portfolio Selection

The analysis so far takes as an input the structure of the portfolio contract and derives the replenishment policy effectively. The portfolio is determined at the beginning at the planning horizon. Thus, in the previous section, the fixed (reservation) cost associated with the contracts could be ignored. It remains to establish, however, the structure of the optimal contract that will minimize both replenishment and reservation cost. This problem is a design problem in which at the beginning of the planning horizon the buyer needs to determine how much capacity to reserve at different suppliers for future time periods.

5.1 General case

We first need to define formally what contracts the supplier is offering the manufacturer. We denote by $\mathcal{X}$ the feasible set for capacities

$$\mathbf{x} = \left(x_t^{i_t}\right)_{t=1,\dots,T}^{i_t=1,\dots,n_t}$$

that the manufacturer can select at period $t = 0$.

Assumption 10 *The feasible set for capacities, $\mathcal{X}$, is a convex set.*

For instance, this assumption captures the case when $\mathcal{X}$ is linearly constrained. We motivate this assumption by presenting some examples of industrial constraints that determine the feasible set of capacities. In all of them, the convexity assumption is satisfied.

Example 2 *Consider a single time period and n options. The supplier offers the manufacturer no more that a certain amount u^i of type i option for every $i = 1, \dots, n$. Then, $\mathcal{X} = \left\{\mathbf{x} \in \mathbb{R}^n_+ | \ x^i \leq u^i, i = 1, \dots, n\right\}$, and $\mathcal{X}$ is convex.*

Example 3 *Consider the single period case. Assume that the supplier offers one flexibility contract with flexibility α and unit cost u: if the manufacturer reserves $x \geq 0$ units in period $t = 0$, then it can order in period $t = 1$ an amount q at a cost of:*

$$r_{flex}(q) = \begin{cases} u(1-\alpha)x & \text{for } q \leq (1-\alpha)x, \\ uq & \text{for } (1-\alpha)x \leq q \leq (1+\alpha)x. \end{cases}$$

On the other hand, consider a portfolio contract made up of 2 options, the first one having terms $v_1 = u$, $w_1 = 0$ (long-term) and $x_1 = (1-\alpha)x$, and the second $v_2 = 0$, $w_2 = u$ (pure option) and $x_2 = 2\alpha x$. The corresponding procurement cost is

$$r_{portf}(q) = \begin{cases} ux_1 & \text{for } q \leq x_1, \\ ux_1 + u(q - x_1) & \text{for } x_1 \leq q \leq x_1 + x_2. \end{cases}$$

We clearly see that any flexibility contract can be replicated by a portfolio made up of these two options with respective capacities, $x_1 \geq 0$ and $x_2 \geq 0$, being constrained by

$$\frac{x_1}{1-\alpha} = \frac{x_2}{2\alpha}.$$

Thus, selecting the optimal level for the flexibility contract is equivalent to finding the optimal portfolio of options 1 and 2 in the following convex feasible set

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^2_+ | \ \frac{x_1}{1-\alpha} = \frac{x_2}{2\alpha} \right\}.$$

Hence, this transformation shows how to include flexibility contracts in the portfolio framework.

Example 4 *In some industries, manufacturers have the policy to purchase a given total amount of capacity for certain groups of suppliers, such as local or minority-owned suppliers. Denote by G the set of such suppliers and γ their target share of business. The policy can then be modeled as the following constraint.*

$$\sum_{i \in G} x^i \geq \gamma \sum_{i=1}^{n} x^i.$$

Such constraint is linear and hence $\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^n_+ | \ \sum_{i \in G} x^i \geq \gamma \sum_{i=1}^{n} x^i \right\}$ *is convex.*

We now define the shape of this up-front cost.

Assumption 11 *Selecting capacities $\mathbf{x} \in \mathcal{X}$ at period $t = 0$ has a cost $v(\mathbf{x})$, where $v(\cdot)$ is a convex function.*

For instance, a typical special case of this up-front cost occurs when $v(\cdot)$ is linear:

$$v(\mathbf{x}) = \sum_{t=1}^{T} \sum_{i_t=1}^{n_t} v_t^{i_t} x_t^{i_t}$$

Of course, this assumption is not satisfied when volume discounts are available on the reservation cost. In this case, one can break the non-convex reservation cost into linear parts by introducing integral variables denoting the type of discount obtained. This makes the feasible set non-convex. Yet, the portfolio selection problem can be broken down into several problems, unfortunately a number exponential with the number of discount levels, where the feasible set is convex and the reservation cost $v(\cdot)$ is linear.

Below we show that the problem of selecting the appropriate portfolio, which is solved at the beginning of the planning horizon, is a concave maximization problem.

Theorem 2 *Given Φ_t, $\forall t = 1, \ldots, T+1$, $V_t(I_t, \Phi_t, \mathbf{x})$ is concave in $(I_t, \mathbf{x})$.*

Corollary 1 *Under Assumption 11, at period $t = 0$ the problem of choosing the capacities, i.e.,*

$$\max_{\mathbf{x} \in \mathcal{X}} \Big\{ \mathbb{E}_{\Phi_1} V_1(0, \Phi_1, \mathbf{x}) - v(\mathbf{x}) \Big\},$$

is a concave maximization problem.

This result is key from a practical standpoint. Indeed, an extensive class of methods from Nonlinear Programming can handle this type of concave maximization problems. Among those, gradient methods, Newton's method or interior point methods, are very well known. It is worth pointing out that in certain cases, e.g., the single period model, we are able to derive closed-form solutions for the derivative of the objective function, in which case the optimization problem is greatly simplified.

5.2 Single period model

Consider the single period model, i.e., $T = 1$. This special case is important since it captures the realities of industries where no inventories are kept. This happens for non-storable components such as electricity or perishable products, or also for systems where inventory storage costs are so high that it is not economically viable to keep inventories.

Assume that the spot market offers unlimited supply, i.e., $\kappa_1 = \infty$ with probability 1. Assume also that

$$v(\mathbf{x}) = \sum_{i=1}^{n} v^i x^i.$$

We sort the available options by execution price, i.e., $w^1 \leq \ldots \leq w^n$, and without loss of generality we assume that $w^n \leq p$. We define $v^{n+1} = 0$ and $w^{n+1} = p$ and for $i = 1, \ldots, n$,

$$y^i = \sum_{k=1}^{i} x^k.$$

In what follows we call D the demand and S the spot market price. Assume that $\mathbb{E}D < \infty$, which implies that all expected profits must be finite. Finally, define

$$J(\mathbf{y}) = -v(\mathbf{x}) + \mathbb{E}_{(D,S)} V_1(0, (D,S), \mathbf{x}).$$

Theorem 3 *With this notation, when J is differentiable, we have that for all $i = 1, \ldots, n$*

$$\frac{dJ}{dy^i} = -v^i + v^{i+1} + \mathbb{E}\Big\{ 1_{y^i \leq D} \Big[\min(S, w^{i+1}) - \min(S, w^i) \Big] \Big\}.$$

The theorem thus provides a closed form equation that characterizes the amounts of options to be purchased by the manufacturer in an optimal portfolio. In particular, as observed in the next section, a similar analysis allows to identify which options are most attractive to the manufacturer and which ones are dominated and should not be considered.

5.3 Attractive option contracts

So far we have shown that the portfolio selection problem is a concave maximization problem. Unfortunately, while this result is numerically important, it does not provide insight about the optimal amounts of options to be purchased by the manufacturer, or insight about the options that the manufacturer should not even consider.

Thus, in this section our objective is to provide conditions and guidelines that help establish whether a specific option is attractive for the manufacturer, i.e. such an option should be part of the optimal portfolio. This is presented in the following propositions.

Proposition 4 *At time period t, option i_t should not be included in an optimal portfolio when*

(i) there is an option k_t such that $v_t^{i_t} > v_t^{k_t}$ and $v_t^{i_t} + w_t^{i_t} > v_t^{k_t} + w_t^{k_t}$;

(ii) there are options j_t and k_t such that $w_t^{j_t} < w_t^{i_t} < w_t^{k_t}$ and

$$\frac{w_t^{i_t} - w_t^{j_t}}{w_t^{k_t} - w_t^{j_t}} v_t^{j_t} + \frac{w_t^{k_t} - w_t^{i_t}}{w_t^{k_t} - w_t^{j_t}} v_t^{k_t} < v_t^{i_t};$$

The proposition thus presents conditions under which a specific option contract can be ignored from an optimal portfolio. As can be seen in the proof of the proposition, this is done by identifying other option contracts that can replace this specific contract and yield a better expected profit.

So far we have identified option contracts that in some sense are *dominated* by other contracts. The next result characterizes option contracts that are dominated by the spot market.

Proposition 5 *Assume that $\kappa_t = \infty$ with probability 1. Then, at time period t, option i_t should not be included in an optimal portfolio when*

$$\mathbb{E}[S_t - w_t^{i_t}]^+ \leq v_t^{i_t}.$$

Figure 4 illustrates Proposition 4. The figure suggests that all the options that the manufacturer should consider have a reservation price small enough so that the option cannot be dominated by other options. If we were to plot all the option prices in a graph, the non-dominated options would appear in lower envelope of the points plotted.

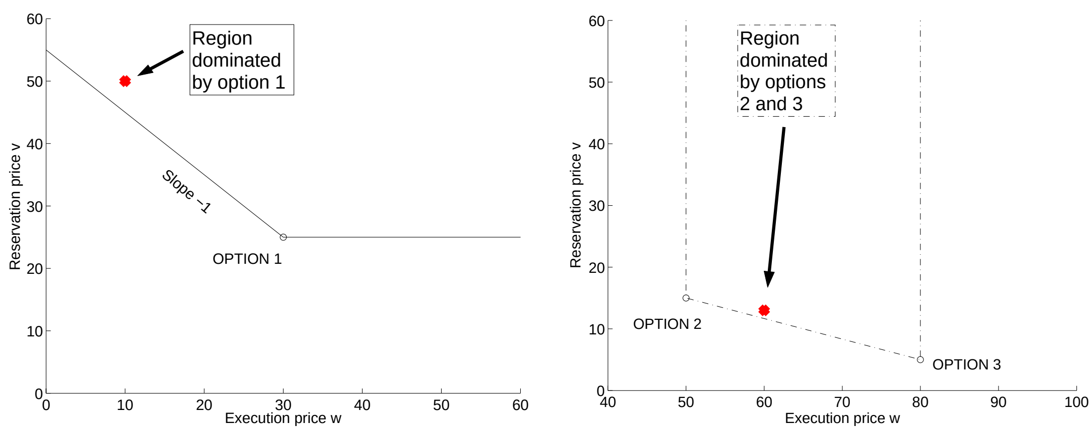

Figure 4: Regions where options are dominated by other options, following the two cases of Proposition 4. The figure on the left illustrates case (i), the figure on the right case (ii).

5.4 Impact on inventory levels

The previous sections provide insights on the value of the different options available to the manufacturer. In this section we present the effects of the portfolio strategy on the inventory target levels.

Theorem 4 *For every time period t, for every source i, $i = 1, \ldots, \overline{n}_t$, the base-stock level b_t^i is such that for every t' and $k = 1, \ldots, n_{t'}$,*

- *for $t' < t$, b_t^i is independent with respect to $x_{t'}^k$;*
- *for $t' = t$, b_t^i is non-decreasing with respect to $x_{t'}^k$;*
- *for $t' > t$, b_t^i is non-increasing with respect to $x_{t'}^k$.*

The intuition behind the theorem is a as follows. The first case is straightforward, since past capacities available at t' should not have an impact on present target inventory levels at $t > t'$. The second case can be explained by observing that the higher the available capacities at a given period the smaller the execution cost at that period. Therefore, it is better to increase the current inventory target levels in order to take advantage of present lower purchasing costs. The third case is also intuitive, it implies that the higher future capacities are at t', the lower present inventories should be at $t < t'$. This is not only true because future purchases reduce the need to currently keep inventory, but also because safety stocks, that protect the manufacturer against situations when demand is higher than the available capacity, can be reduced.

6 Numerical results

6.1 Example solved

Recall the problem posed in Section 2.2. We start by solving the problem when there is no spot market, e.g., the spot market price is very high. In this case, Theorem 3 is sufficient to determine an optimal solution satisfying $\vec{\nabla} J = 0$.

$$\begin{array}{rcccccc} -v^2 + v^1 & = & 4 & = & \mathbb{P}\{y^1 \leq D\}(w^2 - w^1) & = & 6\mathbb{P}\{y^1 \leq D\} \\ -v^3 + v^2 & = & 3 & = & \mathbb{P}\{y^2 \leq D\}(w^3 - w^2) & = & 6\mathbb{P}\{y^2 \leq D\} \\ 0 + v^3 & = & 3 & = & \mathbb{P}\{y^3 \leq D\}(p - w^3) & = & 8\mathbb{P}\{y^3 \leq D\} \end{array}$$

This implies that it is optimal to purchase $x^1_{NoSpot} = 871$, $x^2_{NoSpot} = 129$ and $x^3_{NoSpot} = 96$.

The optimal strategy is different when the spot market becomes competitive, e.g., when the spot market unit price, S, follows a uniform distribution in $[10, 20]$. In this case, it is optimal not to buy any options. That is, the new optimal solution is $x^1_{Spot} = 871$ and $x^2_{Spot} = x^3_{Spot} = 0$.

This implies that in this example the benefit from buying options is undermined by the existence of a spot market. That is, it is better to leave supply uncommitted in order to take advantage of the spot market.

6.2 Sensitivity analysis

In this section, we present computational results that illustrate the sensitivity of the optimal expected profit and corresponding optimal portfolio in terms of the different parameters used in the model. We specifically focus on the impact of the length of the planning horizon, the inventory holding cost, the volatility (standard deviation) of demand and spot market prices and the option reservation price, also known as premium.

The benchmark for the sensitivity analysis consists of a $T = 5$ period model, where $n_t = 2$ identical contracts are offered at every time period. These two contracts are a long-term contract (contract 1) of reservation price 40 and an option contract (contract 2) with premium equal to 20% of the execution price 40, i.e.,

$$v^1 = 40, \quad w^1 = 0,$$
$$v^2 = 8, \quad w^2 = 40.$$

The customer selling price is $p_t = 100$ for every time period $t = 1, \ldots, 5$. In addition, we assume that the demand and the spot market price are independent between them and across time periods, and follow truncated normal distributions with means $\mu_D = 1$ and $\mu_S = 80$, and standard variations $\sigma_D = 0.2$ and $\sigma_S = 20$ respectively. Finally, we assume that the inventory holding cost function is linear and has a unit holding cost equal to $h = 5$.

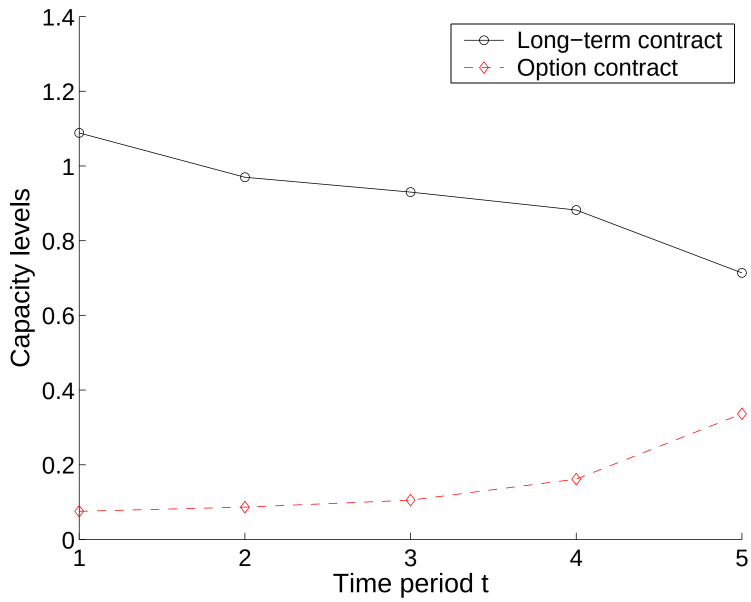

Figure 5: Optimal portfolio for the benchmark case

We observe, in Figure 5, that the optimal portfolio consists of a high capacity level purchased from the long-term contract for every period, and a small level of options. The figure suggests that options are used in increasing amounts as we approach the end of the horizon. A reason for this is that all unused inventory is lost after period 5, and therefore, it is riskier to hold inventory when we approach the end of the selling season. Before period 5, unused inventory can be carried to the next period at a reasonable cost $h = 5$, and as a consequence, flexibility is not needed in great amounts. Finally, we notice that the level of fixed commitment for earlier periods is higher than for later periods. This occurs because we start with no inventory, and thus the manufacturer needs to build some safety stock in the first period.

We compare this benchmark against situations where the parameters, e.g., the length of the planning horizon, inventory holding cost, demand and spot price volatility, option premium, vary. The results are presented in three figures. Figure 6 compares the performance of the optimal portfolio contract to that of relying only on the spot market, i.e., a strategy in which the manufacturer does not have a contract and thus commits no capacity in advance. Figures 7 and 8 depict the optimal levels of the long-term and option contracts as a function of the various parameters.

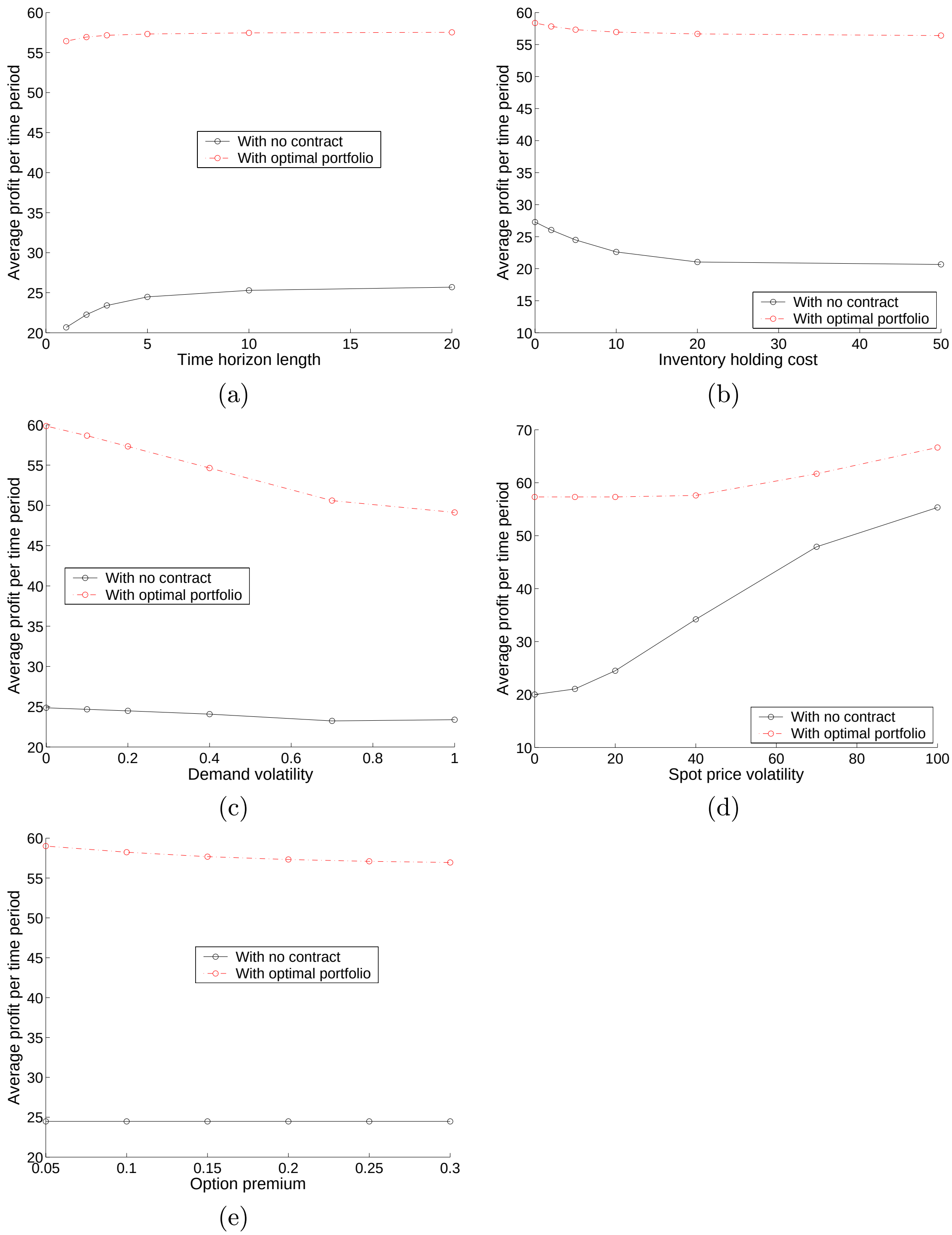

Figure 6: Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the per-period profit, without contract and with the optimal portfolio.

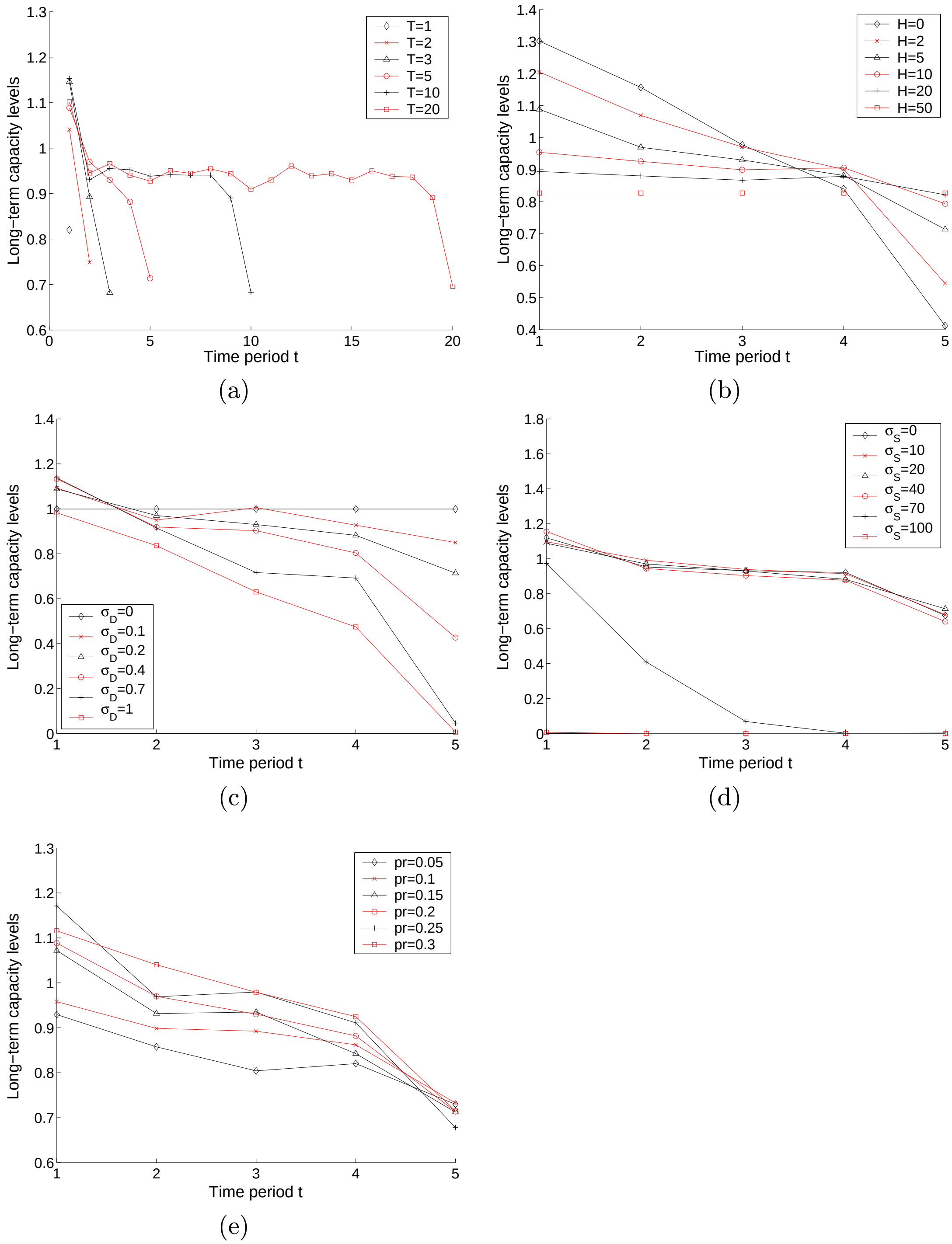

Figure 7: Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the optimal long-term capacity levels.

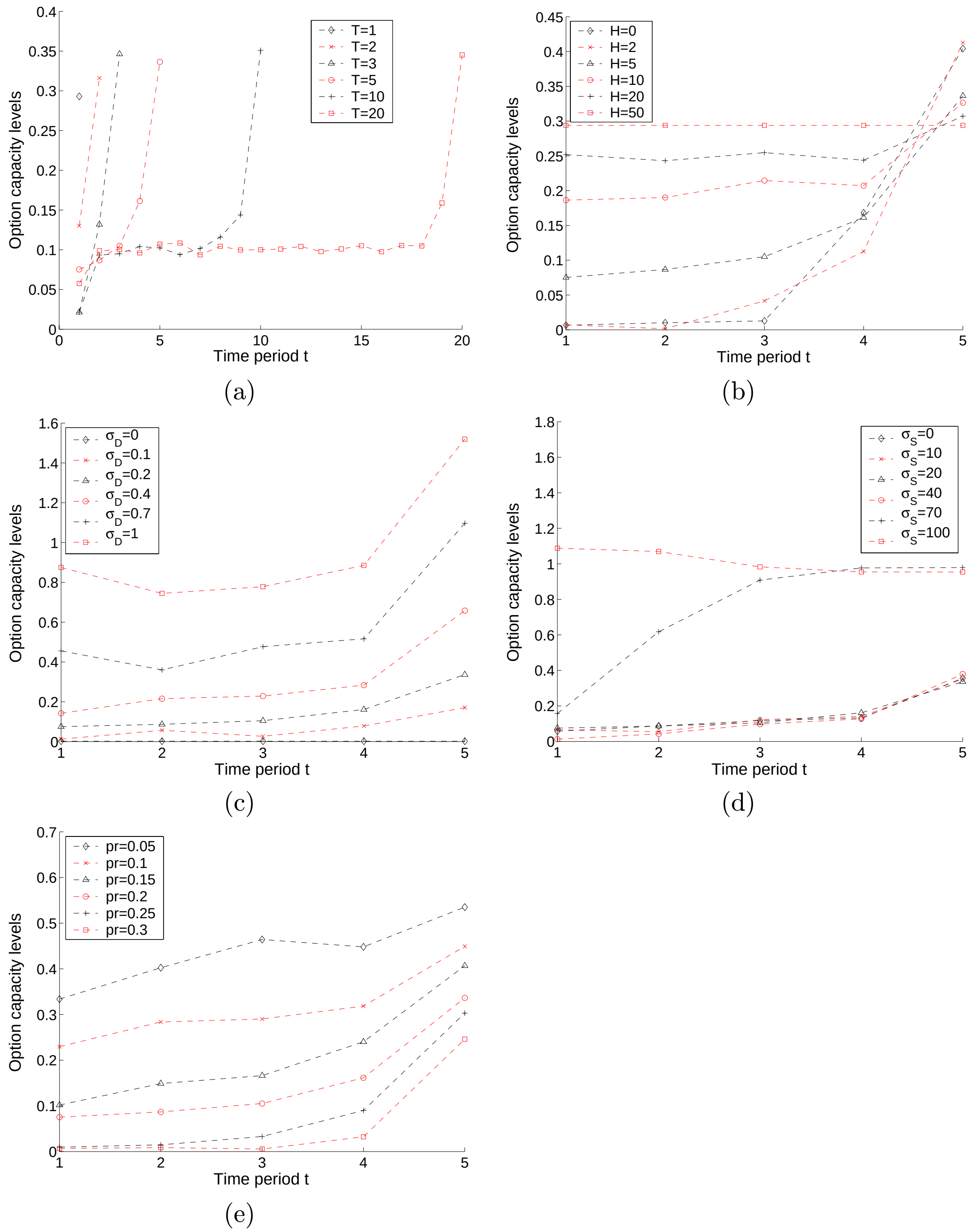

Figure 8: Effect of horizon length (a), inventory cost (b), demand volatility (c), spot price volatility (d) and option premium (e) on the optimal option capacity levels.

Length of planning horizon. We observe in Figure 6(a) that the longer the length of the planning horizon, the higher the expected profit per time period, both under the optimal portfolio or without contracting. This is because it is less risky to hold inventory for longer horizons. Eventually, the per-period profit reaches an upper limit, corresponding to the infinite horizon stationary profit.

Inventory holding cost. When the unit inventory cost is higher, the profits are smaller, both with and without contracts. This is of course intuitive. However, Figure 6(b) also suggests that the portfolio contract is less sensitive to a higher inventory holding cost. This is true since a portfolio strategy allows the manufacturer to rely on future purchases and thus reduces the need for inventory. On the other hand, when the manufacturer relies only on the spot market, it needs to build inventory when prices are attractive and as a result keep higher inventory levels. Finally, Figures 7(b) and 8(b) suggest that, when inventory holding cost is very high, the model can be decoupled into a series of single period models; the level of capacity reserved (long-term and option) in the case of high inventory holding cost is time independent and equal to the single period optimal level.

Demand volatility. In Figure 6(c), we observe that demand volatility, measured by the demand standard deviation, decreases profit, in particular for the portfolio contract case. This is indeed intuitive since, unlike spot purchasing, contracting implies commitments that can be quite expensive but will not necessarily be used. Also, Figures 7(c) and 8(c) suggest that the optimal capacity level is quite sensitive to demand volatility, i.e., the higher the demand uncertainty, the less fixed commitment the manufacturer will make and the higher the option level, exactly what one would expect from a portfolio strategy.

Spot market volatility. The impact of spot market volatility is different than the impact of demand volatility. Indeed, Figure 6(d) shows that the larger the spot price volatility, the larger the expected profits under both the portfolio contract or without any contract. This is true since high variability in spot prices implies opportunities for the manufacturer to stock at a lower price. The impact of spot market volatility is much higher on the no contract strategy. This is because contracting becomes riskier when spot price uncertainty increases, whereas no commitment allows the manufacturer to buy more often at a low spot price and stock for times when spot price is high. Finally, we notice in Figure 8(d) that for higher spot price volatilities, option capacity levels increase dramatically, in contrast to fixed commitment levels, plotted in Figure 7(d). That is, the larger the uncertainty in the spot price, the smaller the level of long-term contracting.

Option premium. We notice that option premium, defined as the ratio of the reservation price and the execution price, does not affect the expected profits of the manufacturer very much, as observed in Figure 6(e). However, while varying the option premium does not affect the manufacturer profit, it has an impact on the optimal portfolio strategy. Indeed, Figures 7(e) and 8(e) show that the higher the option premium, the higher long-term capacity levels and the lower option capacity levels.

A general observation in these experiments is that the optimal long-term capacity levels are decreasing with time. This is intuitive, since the closer we get to the end of the horizon, the larger the risk of being stuck with worthless inventory, and as a consequence, the manufacturer commits to smaller levels. This is illustrated in Figure 7(a). The figure suggests that at the beginning of the horizon, long-term capacity levels are higher in order to build some safety stocks; in the middle of the horizon, the capacity levels are nearly constant; and at the end of the horizon, they decrease abruptly. By contrast, Figure 8(a) suggests reverse dynamics for the option capacity levels. That is, the option level at the beginning of the horizon is relatively small; constant in the middle of the horizon; and much higher at the end of the horizon. This suggests that there is a substitution process between long-term and option levels. That is, the manufacturer prefers long-term contracts at the beginning of the horizon since inventory risk is not high. This is not the case at the end of the planning horizon, where high inventory risk forces the use of option contracts.

We also observe, in Figures 7(a) and 8(a), that the capacity levels in the middle of the horizon are time independent, both for long-term and option contracts. These capacity levels are probably close to those of the stationary levels associated with the infinite horizon model. Thus, for a reasonably long planning horizon, the problem can be broken into three components: a starting transient process in which capacity allows the manufacturer to build inventory; a stationary behavior; and an ending transient process where inventory is disposed at the end of the planning horizon.

6.3 Profit distribution

Our objective in this section is to obtain some insight into the distribution of the manufacturer profit provided by different types of contracts. In this paper, the objective is to maximize expected profit. Decisions based on this objective have an impact not only on the first moment of profit but also on the entire distribution of profit. We investigate here the effects of capacity decisions on the profit distribution.

We show here an instance with two time periods and independent demands D_1 and D_2, following truncated (non-negative) normal distributions with means $\mu_1 = 100$ and $\mu_2 = 200$ and standard deviations $\sigma_1 = 40$ and $\sigma_2 = 100$ respectively. The end-customer prices are $p_1 = p_2 = 15$.

We present the computation of the distribution of profit for three cases:

1. Optimal (for expected profit) long-term contract at unit price $v_1^1 = 9, v_2^1 = 7$ (no execution cost: $w_1^1 = w_2^1 = 0$).
2. Optimal option contract at execution unit price $w_1^2 = 9, w_2^2 = 7$ and up-front unit price $v_1^2 = v_2^2 = 2$.
3. Optimal portfolio of both.

We first selected the optimal contract in terms of expected profit by using a dynamic program. The optimal capacities were:

1. $x_1^1 \approx 120$ and $x_2^1 \approx 180$ for the long-term only contract, yielding an expected profit of $1,517$.
2. $x_1^2 \approx 120$ and $x_2^2 \approx 260$ for the option only contract, yielding $1,281$.
3. $x_1^1 \approx 100$, $x_2^1 \approx 140$, $x_1^2 \approx 20$ and $x_2^2 \approx 110$ for the portfolio contract, yielding $1,613$.

Then we ran a Monte Carlo simulation to estimate the distribution of profit for each of the contracts.

Figure 9 depicts the simulation results. It shows that, as expected, the portfolio contract has the largest expected profit. On the other hand, the coefficient of variation of the profit obtained by the portfolio contract is smaller than that of the long-term contract but larger than that of the option contract. Computational studies with different data sets provide similar results: portfolio contracts dominate long term contracts both from profit and financial risk points of view.

7 Extensions

In this section we report on various extensions of our results.

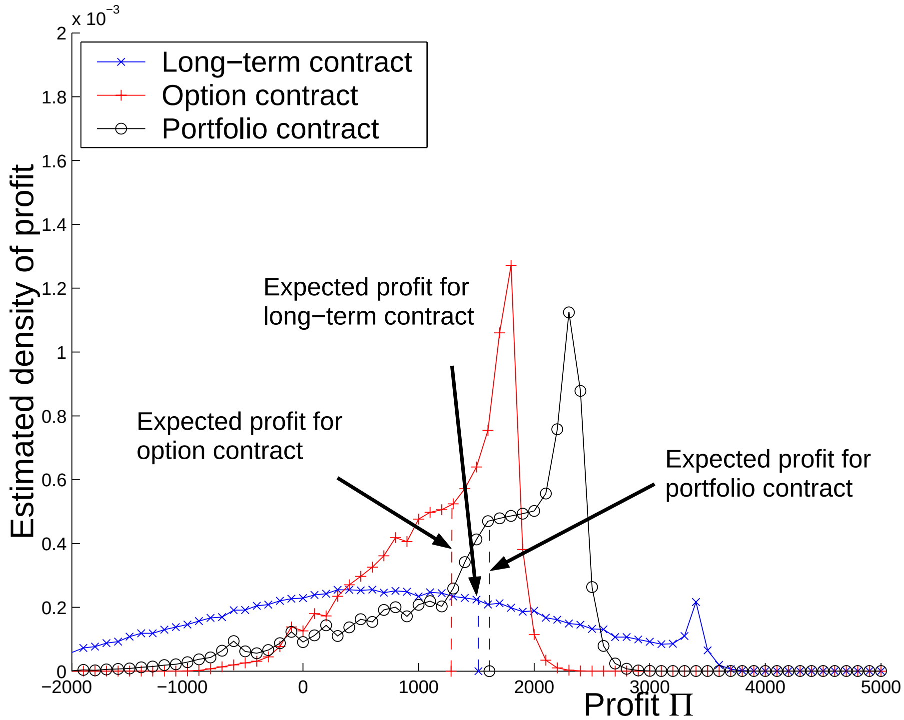

Figure 9: Comparison of profit distribution for different contracts.

7.1 Backlogging Model

It is straightforward to extend the framework developed in this paper to a multi-period newsvendor model, i.e., a model in which each period's order is made before demand is known and all shortages are backlogged. This is done by replacing each period t by periods $(t^a,\ t^b)$, $t = 1, \ldots, T$, with the following representations:

- In period t^a, the manufacturer chooses the best trade-off between buying from the contract, purchased at period 0, purchasing from the spot market at current market conditions, or depleting inventory carried from period to period.
- In period t^b, demand is realized and the manufacturer serves it all with the available inventory.

Notice that, in this formulation, the manufacturer serves demand if it has inventory on hand, and backlogs it otherwise. Decisions are made in periods t^a, $t = 1, \ldots, T$. Specifically, in time t^a one needs to decide on order quantities and spot market purchases; these items will be added to inventory at period t^b.

This sequence of events implies that we can no longer guarantee that $\mathbf{I} \geq 0$. Indeed, customer demand being random, it can always be big enough to bring the inventory position below 0. Therefore, in addition to the new sequence of events, one has to use a backlog model instead of a lost sales model. We thus have to define $h_t(\cdot)$ as both inventory holding and backlogging cost function for each $t = 2, \ldots, T$.

The results obtained in this paper hold under this framework as well. However, one important modification is that the number of available sources of supply is reduced by one since demand is fully served. Namely, in Theorem 1, the number of real sources should be $\overline{n}_t = n_t + 1$ instead of $n_t + 2$. Moreover, if we also remove the spot market, then $\overline{n}_t = n_t$.

7.2 Contracts with total capacity commitment

The model and results described in this paper can be extended to include in the portfolio a *total capacity commitment contract*: the manufacturer purchases at the beginning of the planning horizon capacity to be used anytime during the planning horizon. Thus, at each time period, this special contract is a source of supply with a capacity equal to the initial capacity minus what has already been used. The execution cost in this case might be dependent on the time period. Examples of this type of contracts can be found in [1] or in [4].

The analysis of a portfolio that includes such contracts is similar to what has been presented in the current paper. We can examine the case in which one of the supply sources has a constraint on the total capacity available across time periods. Let l_1 be the initial capacity available at the beginning of the planning horizon.

Of course, we need to add to the space state the amount of capacity left at the beginning of time period t; we denote this amount by l_t. Let u_t be the execution unit cost at period t for any amount q_t less than the remaining capacity, l_t. Under these assumptions the dynamic program, Equation (4), is modified as follows:

$$V_t(I_t, l_t, \Phi_t,) = p_t d_t - h_t(I_t) + \max_{z_t, q_t} -C_t(z_t) - u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t)$$
$$\text{subject to} \begin{cases} z_t \geq 0 \\ q_t \geq 0 \\ I_{t+1} = I_t - d_t + z_t + q_t \geq 0, \end{cases} \tag{7}$$

where U_{t+1} and C_t are defined similarly to Equations (3) and (6). The convexity results presented in Proposition 1 and Corollary 1 still hold by conducting the same induction arguments and using Lemma 1. Of course, the structure of the optimal replenishment policy is slightly different since it depends now not only on I_t but also on l_t. For this purpose, observe that

$$\begin{aligned} &\max_{z_t, q_t} -C_t(z_t) - u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \\ = &\max_{z_t} \Big\{ -C_t(z_t) + \Big[\max_{q_t} -u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \Big] \Big\}. \end{aligned}$$

It is straightforward to see that, given l_t, the optimal control for z_t behaves as described in Theorem 1 and illustrated in Figure 3. This is true since the function

$$\max_{q_t} -u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t)$$

is a concave function of $I_t + z_t$. Note that in this case, the breakpoints characterizing the inventory policy depend on l_t. In this sense, the structure is preserved. Differently, by using now that

$$\begin{aligned} &\max_{z_t, q_t} -C_t(z_t) - u_t q_t + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \\ = &\max_{q_t} \Big\{ -u_t q_t + \Big[\max_{z_t} -C_t(z_t) + U_{t+1}(I_t - d_t + z_t + q_t, l_t - q_t) \Big] \Big\}, \end{aligned}$$

we observe that given l_t, the optimal control for q_t is not necessarily piecewise linear in I_t. Indeed, the reason for this is that using the remaining capacity has an influence not only on future inventory position I_{t+1} but also on future capacity $l_{t+1} = l_t - q_t$.

7.3 Random selling price

Replace Assumption 5 by the following assumption.

Assumption 12 *For $t = 1, \ldots, T$, the price p_t at which the manufacturer charges the end customers is random and is observed at the beginning of period t only. Prior to period t, the distribution of p_t depends only on the information state, and not on the manufacturer's decisions.*

This assumption is commonly satisfied in markets in which the manufacturer is price-taker and the market price depends on external factors such as the state of the economy.

By conducting the same analysis with Assumption 12 instead of Assumption 5, all the results remain valid. Finally, we can also extend our results to a random salvage value a.

7.4 Application to disruption management

When signing option contracts with suppliers, the manufacturer might question the ability of the suppliers to honor deliveries in full. Similar to credit risk in finance, we can define for every time period t and option contract i_t, a random variable $A_t^{i_t}$ in $[0, 1]$ such that the amount that can be ordered by the manufacturer from contract i_t at time t is no more than $x_t^{i_t} A_t^{i_t}$. Thus, the random variable $A_t^{i_t}$ corresponds to the effective portion of capacity that the manufacturer is able to request.

Typically, $A_t^{i_t}$ can be correlated with customer demand and spot market prices, and depends exclusively on the past information. Formally, for $t = 1, \ldots, T$, at the beginning of period t, $A_t^1, \ldots, A_t^{n_t}$ become known (and thus are included in the information vector Φ_t) and, based on these, the manufacturer decides the amount of supply to purchase from every contract (t, i_t), for $i_t = 1, \ldots, n_t$:

$$\text{For } i_t = 1, \ldots, n_t \text{ request amount } 0 \leq q_t^{i_t} \leq a_t^{i_t} x_t^{i_t},$$

where $a_t^{i_t}$ is the outcome of the random variable $A_t^{i_t}$. With this modeling, all the analysis can be conducted without changes, and therefore, the results still hold.

7.5 Models with arbitrage opportunities

In certain industrial situations, the manufacturer may be able to sell back excess inventory to the spot market and thus supply contracts may provide arbitrage opportunities. This is particularly relevant for OEMs, e.g., automotive, PC or electronics manufacturers, since they are natural consumers of commodity components. Thus, manufacturers in these industries could take advantage of their size in order to realize additional profits from pure trading. In this setting, financial models may not be applicable to industrial procurement; this is true since procurement spot markets are far from being efficient, due to the limited number of suppliers and buyers or operational characteristics, e.g., lead times and transaction costs. Our framework provides a good starting point to model capacitated spot markets and hence explore these issues. The model can indeed handle the possibility of selling back to the spot market, provided that the spot market cost function is convex and that arbitrage opportunities are limited to finite amounts (similarly to Assumption 7). Under this assumption, all the results provided in Section 3 hold. In addition, the results of Sections 4 and 5 hold if we modify Assumption 8 to allow sales to the spot market, as long as the unit sale price is constant smaller than the spot market unit purchase price.

8 Conclusion

The ability to establish effective supply contracts has been recently recognized, both in industry and academia, as an important contributor to the success of the firm. Indeed, effective supply contracts play an important role in manufacturers' abilities to reduce cost and ensure adequate supply of components. This is especially important for products for which customer demand is highly uncertain and hence supply flexibility is necessary.

Identifying effective contracts for commodity products is even more challenging since the manufacturer typically has a number of options to choose from: many suppliers willing to sign long-term or option contracts as well as spot market purchasing. Thus, portfolio contracts may be appropriate as they provide flexibility over long-term contracts, and increase expected profit by effectively selecting the various options taking into account uncertainty in the spot market.

The paper thus develops a general framework for the design of supply contracts in a multi-period environment. This framework is especially suited for the analysis of a portfolio of different options. In this case, the optimal replenishment policy is particularly simple: *every option's optimal execution policy is a modified base-stock policy.* Moreover, the contract design problem has a concave structure which makes it tractable in practice. In certain cases, we can derive closed-form formulas and find an analytical expression for the optimal solution.

At this point it is important to point out an important limitation of our model. So far, we have focused on the problem of maximizing expected profit. One might wonder, however, if it is possible to modify the model to take risk explicitly into account. While this is an important challenge, it is appropriate to point out that the limited numerical study conducted in Section 6.3 suggests that the portfolio contracts designed by our model reduce risks relative to other contracts. This issue is discussed in a companion paper.

Finally, another important question is why suppliers would agree to sell option contracts and not insist on firm commitments, since they will incur the cost of reserving capacity. This question was in part addressed by several authors, see [5] or [6], who showed that a portfolio of an option and a long-term contract is profitable to both players, the manufacturer and supplier: the arrangement is "win-win" rather than a zero-sum game. More importantly, since a single supplier typically faces several manufacturers, the supplier is better positioned to handle demand uncertainty due to the risk pooling effect. Thus, the suppliers' trade-offs are clear. On the one hand, long-term contracts guarantee a certain revenue but result in smaller order quantities from manufacturers due to lack of flexibility. On the other hand, option contracts provide suppliers with higher margins but uncertain revenue. This suggests the following challenge. Is it possible to develop a model that allows suppliers to offer an optimal menu of options so that the supplier's profit is maximized? This issue is addressed in a different paper.

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A Proofs

A.1 Lemma 1

Proof. Let $b_1, b_2 \in Q$ and $\lambda \in (0,1)$. Define $b_3 = \lambda b_1 + (1-\lambda)b_2$. Since $b_1, b_2 \in Q$, $P(b_1)$ and $P(b_2)$ are not empty, so we can find $x_1 \in P(b_1)$ and $x_2 \in P(b_2)$. Hence, $x_3 = \lambda x_1 + (1-\lambda)x_2 \in P(b_3)$ and thus $P(b_3) \neq \emptyset$. Thus, $b_3 \in Q$ which implies that Q is a convex set.

We now show that $g(\cdot)$ is concave on Q. For this purpose, consider two different cases depending on whether $g(b_1)$ and $g(b_2)$ are finite, or one of them is infinite.

If $g(b_1) < \infty$ and $g(b_2) < \infty$, take $\epsilon > 0$. We can find close-to-optimum values $x_1 \in P(b_1)$ and $x_2 \in P(b_2)$ such that $f(x_1, b_1) \geq g(b_1) - \epsilon$ and $f(x_2, b_2) \geq g(b_2) - \epsilon$. Clearly, $x_3 = \lambda x_1 + (1 - \lambda)x_2 \in P(b_3)$ and thus $g(b_3) \geq f(x_3, b_3)$. Moreover, the concavity of $f(\cdot)$ implies that $f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda) f(x_2, b_2)$. Hence,

$$g(b_3) \geq f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda) f(x_2, b_2) \geq \lambda g(b_1) + (1 - \lambda) g(b_2) - \epsilon.$$

Since this is true for every $\epsilon > 0$, we must have $g(b_3) \geq \lambda g(b_1) + (1 - \lambda) g(b_2)$.

Consider the second case in which either $g(b_1)$ or $g(b_2)$ is infinite. Without loss of generality, assume that $g(b_1) = \infty$. Then, for every $M > 0$, we can find $x_1 \in P(b_1)$ such that $f(x_1, b_1) \geq M$. Fix $x_2 \in P(b_2)$. Then again $x_3 = \lambda x_1 + (1 - \lambda)x_2 \in P(b_3)$ and $g(b_3) \geq f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda) f(x_2, b_2)$. Hence,

$$g(b_3) \geq f(x_3, b_3) \geq \lambda f(x_1, b_1) + (1 - \lambda) f(x_2, b_2) \geq \lambda M + (1 - \lambda) f(x_2, b_2).$$

Since this is true for every $M > 0$, and since $\lambda > 0$, we have $g(b_3) = \infty$.

In any case, $g(b_3) \geq \lambda g(b_1) + (1 - \lambda) g(b_2)$, so $g(\cdot)$ is concave over Q. ■

A.2 Proposition 1

Proof. We prove the property by induction on t starting at period $T + 1$ and moving backwards to the first period. For $t = T + 1$, $V_{T+1}(\cdot, \Phi_{T+1})$ is a linear function of I_{T+1} and thus concave.

Assume now that the statement is true for $t + 1$: given Φ_{t+1}, $V_{t+1}(\cdot, \Phi_{t+1})$ is concave in I_{t+1}. We show the inductive proposition for t. For this purpose, we assume that Φ_t is given, and hence the values of d_t and $s_t(\cdot)$ are known. Since $\Phi_{t+1} = (\Phi_t, s_{t+1}, d_{t+1})$ is a random variable that depends only on Φ_t, the distribution of Φ_{t+1} is also known.

Given Φ_{t+1}, the expected profit-to-go in period $t + 1$, $V_{t+1}(\cdot, \Phi_{t+1})$, is concave in I_{t+1}. Let $U_{t+1}(I_{t+1}) = \mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1})$ which is also concave in I_{t+1}. Thus, we can rewrite V_t as follows:

$$V_t(I_t, \Phi_t) = \quad p_t I_t - h_t(I_t) + \max_{q_t^r, q_t^s, I_{t+1}} p_t(-I_{t+1} + q_t^r + q_t^s) - r_t(q_t^r) - s_t(q_t^s) + U_{t+1}(I_{t+1})$$

$$\text{subject to} \begin{cases} -q_t^r \leq 0 \\ -q_t^s \leq 0 \\ -I_{t+1} \leq 0 \\ -I_{t+1} + q_t^r + q_t^s \leq I_t \\ I_{t+1} - q_t^r - q_t^s \leq d_t - I_t \end{cases}$$

The maximization problem is constrained by linear inequalities involving (q_t^r, q_t^s, I_{t+1}). Also, the objective function,

$$p_t(-I_{t+1} + q_t^r + q_t^s) - r_t(q_t^r) - s_t(q_t^s) + U_{t+1}(I_{t+1}),$$

is jointly concave in the control variables, since it is the sum of a linear function and three concave functions. Moreover, for any value of $I_t \geq 0$, the feasible set is not empty because $q_t^r = q_t^s = 0$ and $I_{t+1} = I_t \geq 0$ belongs to it. By Lemma 1, this objective is jointly concave in $d_t - I_t$ and I_t. Hence, it is concave in I_t.

Finally, since $V_t(I_t, \Phi_t)$ is the sum of a linear function and two functions that are concave in I_t, we have that $V_t(\cdot, \Phi_t)$ is concave. ■

A.3 Proposition 2

Proof. Consider the replenishment problem defined by Equation (4). This problem implies that we are maximizing a concave function over the interval $[\max(0, I_t - d_t), \infty)$. Since the feasible set is linearly constrained, the Karush-Kuhn-Tucker (KKT) conditions hold: the optimal solution is I^*_{t+1} if and only if there exists dual multiplier $\mu \geq 0$ such that

$$\mu \in \partial C_t(I^*_{t+1} - I_t + d_t)$$
$$\mu \in \partial U_{t+1}(I^*_{t+1}).$$

Consider $I^0_t \geq 0$ and $I^1_t \geq 0$ such that $I^1_t > I^0_t$. Let $I^0_{t+1} = I^*_{t+1}(I^0_t)$ and let μ^0 be an optimal multiplier satisfying the corresponding KKT conditions. Also, let $I^1_{t+1} = I^*_{t+1}(I^1_t)$ and μ^1 be an optimal multiplier.

Assume that $I^1_{t+1} < I^0_{t+1}$. Then, since μ^1 belongs to the sub-gradient of C_t at $I^1_{t+1} - I^1_t + d_t$ and $I^1_{t+1} - I^1_t + d_t < I^0_{t+1} - I^0_t + d_t$, we must have that $\mu^1 \leq \mu^0$, because C_t is convex. Moreover, since $\mu^1 \in \partial U_{t+1}(I^1_{t+1})$, $\mu^0 \in \partial U_{t+1}(I^0_{t+1})$ and $I^1_{t+1} < I^0_{t+1}$, since U_{t+1} is concave, we must have $\mu^1 \geq \mu^0$. Hence, $\mu^1 = \mu^0$.

Finally, by construction

$$I^0_{t+1} + d_t - I^0_t > I^1_{t+1} + d_t - I^0_t > I^1_{t+1} + d_t - I^1_t,$$
$$\mu^0 \in \partial C_t(I^0_{t+1} - I^0_t + d_t)$$

and

$$\mu^1 \in \partial C_t(I^1_{t+1} - I^1_t + d_t).$$

Hence, the convexity of C_t, together with $\mu^1 = \mu^0$, implies that

$$\mu^1 \in \partial C_t(I^1_{t+1} - I^0_t + d_t).$$

Therefore, $\mu^1 \in \partial U_{t+1}(I^1_{t+1}) \cap \partial C_t(I^1_{t+1} - I^0_t + d_t)$ so that by the KKT conditions I^1_{t+1} is an optimal control for the inventory position I^0_t too. Since I^0_{t+1} was the smallest optimal control at I^0_t, we must have $I^1_{t+1} \geq I^0_{t+1}$ which is a contradiction. So, $0 \leq I^*_{t+1}(I^1_t) - I^*_{t+1}(I^0_t)$.

Assume now that $I^1_{t+1} > I^0_{t+1} + I^1_t - I^0_t$. Then $I^1_{t+1} - I^1_t + d_t > I^0_{t+1} - I^0_t + d_t$, and since C_t is convex, we must have $\mu^1 \geq \mu^0$. Similarly, since $I^1_{t+1} > I^0_{t+1}$ and U_{t+1} is concave, $\mu^1 \leq \mu^0$. So again $\mu^1 = \mu^0$.

But in this case, we have that $I^0_{t+1} < I^0_{t+1} + I^1_t - I^0_t < I^1_{t+1}$. By repeating the previous argument, $\mu^1 \in \partial U_{t+1}(I^0_{t+1} + I^1_t - I^0_t) \cap \partial C_t(I^0_{t+1} - I^0_t + d_t)$. Hence, $I^0_{t+1} + I^1_t - I^0_t$ is an optimal control for the inventory position I^1_t, which is a contradiction because I^1_{t+1} was supposed to be the smallest optimal control for I^1_t. Consequently, $I^*_{t+1}(I^1_t) - I^*_{t+1}(I^0_t) \leq I^1_t - I^0_t$. ■

A.4 Proposition 3

Proof. Consider I_t such that $\left.\frac{d^2 C_t}{dz_t^2}\right|_{I^*_{t+1}(I_t) - I_t + d_t}$ and $\left.\frac{d^2 U_{t+1}}{dI_{t+1}^2}\right|_{I^*_{t+1}(I_t)}$ are well defined, i.e., they are twice differentiable at I_t. Assume too that

$$\left.\frac{d^2 C_t}{dz_t^2}\right|_{I^*_{t+1}(I_t) - I_t + d_t} - \left.\frac{d^2 U_{t+1}}{dI_{t+1}^2}\right|_{I^*_{t+1}(I_t)} \neq 0.$$

Since U_{t+1} and C_t are twice differentiable at $I^*_{t+1}(I_t)$ and $I^*_{t+1}(I_t) - I_t + d_t$ respectively, and I^*_{t+1} is differentiable at I_t, then for $\epsilon > 0$ small enough, U_{t+1} and C_t are (once) differentiable at $I^*_{t+1}(I_t + \delta)$ and $I^*_{t+1}(I_t + \delta) - I_t - \delta + d_t$, respectively, for any δ such that $-\epsilon \leq \delta \leq \epsilon$.

In the differentiable case, the KKT optimality conditions can be simplified and written as

$$\frac{dU_{t+1}}{dI_{t+1}}_{|I^*_{t+1}(I_t+\delta)} = \frac{dC_t}{dz_t}_{|I^*_{t+1}(I_t+\delta)-I_t-\delta+d_t}.$$

This equality holds for $-\epsilon \leq \delta \leq \epsilon$. Moreover, since we assume I^*_{t+1} to be differentiable at I_t, the left-hand side and the right-hand side of the above equation are differentiable at $\delta = 0$ and the derivative of the left-hand and right-hand terms must be equal at $\delta = 0$. By the chain rule,

$$\frac{d^2U_{t+1}}{dI^2_{t+1}}_{|I^*_{t+1}(I_t)} \frac{dI^*_{t+1}}{dI_t}_{|I_t} = \frac{d^2C_t}{dz_t^2}_{|I^*_{t+1}(I_t)-I_t+d_t} \left(\frac{dI^*_{t+1}}{dI_t}_{|I_t} - 1\right).$$

We can rearrange this expression into

$$\frac{dI^*_{t+1}}{dI_t}_{|I_t} = \frac{\frac{d^2C_t}{dz_t^2}_{|I^*_{t+1}(I_t)-I_t+d_t}}{\frac{d^2C_t}{dz_t^2}_{|I^*_{t+1}(I_t)-I_t+d_t} - \frac{d^2U_{t+1}}{dI^2_{t+1}}_{|I^*_{t+1}(I_t)}}.$$

■

A.5 Theorem 1

Proof. The proof of Proposition 2 tells us that the objective function is concave and the feasible set is convex. As result the optimality condition is

$$\exists \mu \geq 0 \text{ such that } \mu \in \partial C_t(I_{t+1} - I_t + d_t) \text{ and } \mu \in \partial U_{t+1}(I_{t+1}).$$

Equation (6) implies that

$$\partial C_t(I_{t+1} - I_t + d_t) = \begin{cases} \{\overline{w}_k\} & \text{when } \sum_{j=1}^{k-1} \overline{x}_j < I_{t+1} - I_t + d_t < \sum_{j=1}^{k} \overline{x}_j \\ [\overline{w}_k, \overline{w}_{k+1}] & \text{when } I_{t+1} - I_t + d_t = \sum_{j=1}^{k} \overline{x}_j. \end{cases}$$

Hence, we can use this structure and optimality conditions to characterize an optimal replenishment strategy. We can define break-points $f_0 \leq \ldots \leq f_{2\overline{n}_t}$ that satisfy the following description.

- If $\mu = \overline{w}_k$ for some $k = 1, \ldots, \overline{n}_t$, then $\overline{w}_k \in \partial U_{t+1}(I_{t+1})$ determines a fixed I_{t+1}, because we adopted the convention that we always pick the smallest optimal I_{t+1}. This case happens when $\sum_{j=1}^{k-1} \overline{x}_j \leq I_{t+1} - I_t + d_t \leq \sum_{j=1}^{k} \overline{x}_j$ which is equivalent to $I_t \in [f_{2k-1}, f_{2k}]$.
- Otherwise there is $k = 0, \ldots, \overline{n}_t - 1$ such that $\overline{w}_k < \mu < \overline{w}_{k+1}$. In that case, the order quantity $I_{t+1} - I_t + d_t = \sum_{j=1}^{k} \overline{x}_j$, which is a constant. Observe that μ belongs to the interval $[\overline{w}_k, \overline{w}_{k+1}]$ for a given range of I_{t+1}. This is true since $\mu \in \partial U_{t+1}(I_{t+1})$. This, together with the fact that $I_{t+1} - I_t$ is fixed, implies that there also is a range $[f_{2k}, f_{2k+1}]$ for I_t for which $\mu \in [\overline{w}_k, \overline{w}_{k+1}]$.

All these intervals are adjacent and alternate. They cover $\mathbb{R}_+$, which implies that $f_0 = 0$ and $f_{2\overline{n}_t} = \infty$.

We can now derive the optimal replenishment policies from this characterization.

Interval	$[f_{2k-1}, f_{2k}]$	$[f_{2k}, f_{2k+1}]$
I_{t+1}	constant	increasing with slope 1
z_t	decreasing with slope -1	constant

Clearly, the manufacturer will replenish inventory starting with the least expensive source, i.e., suppliers or spot market, and moving to more expensive ones as capacity is exhausted. Thus, for $i = 1, \ldots, \overline{n}_t$, let $b_t^i = f_{2i}$ and observe that b_t^i is the base-stock level for source i. Moreover, since we start using source $i+1$ when source i is exhausted, we must have that $b_t^i \leq b_t^{i+1} - \overline{x}_t^i$. ■

A.6 Theorem 2

Proof. We prove the property by induction from $t = T+1$ to $t = 1$. The proof is similar to the proof of Proposition 1. For $t = T+1$, $V_{T+1}(I_{T+1}, \Phi_{T+1}, \mathbf{x}) = aI_{T+1}$ and is thus concave in $(I_{T+1}, \mathbf{x})$.

Assume now that the lemma is true for $t+1$: given Φ_{t+1}, $V_{t+1}(I_{t+1}, \Phi_{t+1}, \mathbf{x})$ is concave in $(I_{t+1}, \mathbf{x})$. We show the inductive proposition for t. For this purpose, we assume that Φ_t is given, and therefore d_t, c_t, κ_t and the distribution of Φ_{t+1} are fixed.

Clearly, since given Φ_{t+1}, $V_{t+1}(I_{t+1}, \Phi_{t+1}, \mathbf{x})$ is concave in $(I_{t+1}, \mathbf{x})$, we must have that $U_{t+1}(I_{t+1}, \mathbf{x}) = \mathbb{E}_{\Phi_{t+1}} V_{t+1}(I_{t+1}, \Phi_{t+1}, \mathbf{x})$ is concave in $(I_{t+1}, \mathbf{x})$.

Combining Equations (4) and (6) we have,

$$V_t(I_t, \Phi_t, \mathbf{x}) = -h_t(I_t) + p_t d_t + \max_{q_t^1, \ldots, q_t^{\overline{n}_t}, I_{t+1}} - \sum_{k=1}^{\overline{n}_t} \overline{w}_t^k q_t^k + U_{t+1}(I_{t+1}, \mathbf{x})$$

$$\text{subject to} \begin{cases} -I_{t+1} \leq 0 & \\ -q_t^k \leq 0 & \forall k = 1, \ldots, \overline{n}_t \\ q_t^k \leq \overline{x}_t^k & \forall k = 1, \ldots, \overline{n}_t \\ \sum_{k=1}^{\overline{n}_t} q_t^k - I_{t+1} = d_t - I_t & \end{cases}$$

Observe that there is always a feasible solution to this maximization problem; for any $I_t \geq 0$: $I_{t+1} = I_t$, $q_t^i = 0$ for $i = 1, \ldots, \overline{n}_t$, except for the source of not serving demand, that we set to d_t, is a feasible solution. To apply Lemma 1 note also that the objective of the above maximization problem is concave in $(q_t^1, \ldots, q_t^{\overline{n}_t}, I_{t+1}, \overline{\mathbf{x}}, d_t - I_t, I_t - d_t)$. In addition, we can rewrite the feasible region of the maximization problem as follows:

$$A \begin{pmatrix} q_t^1 \\ \vdots \\ q_t^{\overline{n}_t} \\ I_{t+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \overline{\mathbf{x}} \\ d_t - I_t \\ I_t - d_t \end{pmatrix}$$

where A is a matrix of $0, 1, -1$.

Thus, Lemma 1 tells us that the objective of the maximization problem is concave in $(\overline{\mathbf{x}}, d_t - I_t, I_t - d_t)$, and hence in $(I_t, \mathbf{x})$. Therefore, since $V_t(\cdot, \Phi_t, \cdot)$ is the sum of concave functions, it is concave as in $(I_t, \mathbf{x})$. ■

A.7 Theorem 3

Proof. Given $\mathbf{x}$, we define $\mathbf{y}$ by

$$y^i = \sum_{k=1}^{i} x^k, i = 1, \dots, n.$$

The term $v(x)$ can be written as

$$\sum_{i=1}^{n} v^i x^i = \sum_{i=1}^{n} (v^i - v^{i+1}) y^i$$

by defining $v^{n+1} = 0$. Define also $w^{n+1} = p$ and $y^{n+1} = \infty$.

We analyze the profit for every sample path. If $w^{i_0} \leq S \leq w^{i_0+1}$ for some $i_0 = 1, \dots, n$, the profit that we obtain is

$$\begin{aligned}
\Pi = & -\sum_{i=1}^{n} (v^i - v^{i+1}) y^i + \sum_{i=1}^{i_0} (p - w^i) \Big[\min(D - y^{i-1}, x^i) \Big]^+ + (p - S) \Big[D - y^{i_0} \Big]^+ \\
= & -\sum_{i=1}^{n} (v^i - v^{i+1}) y^i + \sum_{i=1}^{i_0} (p - w^i) \Big[\min(D, y^i) - \min(D, y^{i-1}) \Big] + (p - S) \Big[\min(D, y^{i_0+1}) - \min(D, y^{i_0}) \Big] \\
= & -\sum_{i=1}^{n} (v^i - v^{i+1}) y^i + \sum_{i=1}^{i_0+1} \Big[p - \min(S, w^i) \Big] \Big[\min(D, y^i) - \min(D, y^{i-1}) \Big],
\end{aligned}$$

by remarking that $\min(D - y^{i-1}, x^i) = \min(D, y^i) - y^{i-1}$, and that $[\min(D, y^i) - y^{i-1}]^+ = \min(D, y^i) - \min(D, y^{i-1})$. Using the fact that

$$p - \min(S, w^i) = \sum_{j=i}^{n} (\min(S, w^{j+1}) - \min(S, w^j)),$$

we can rearrange this expression into

$$\Pi = \quad -\sum_{i=1}^{n} (v^i - v^{i+1}) y^i + \sum_{i=1}^{n} \Big[\min(S, w^{i+1}) - \min(S, w^i) \Big] \min(D, y^i).$$

The same equation is of course true when $S > p$. We can now take expectation on (D, S) and the fact that $\mathbb{E}D < \infty$ guarantees that this expectation is well defined. Hence,

$$\mathbb{E}_{(D,S)} \Pi = J = -\sum_{i=1}^{n} (v^i - v^{i+1}) y^i + \sum_{i=1}^{n} \mathbb{E} \Big\{ \Big[\min(S, w^{i+1}) - \min(S, w^i) \Big] \min(D, y^i) \Big\}.$$

After taking the derivative with respect to y^i, we obtain that,

$$\frac{dJ}{dy^i} = v^{i+1} - v^i + \mathbb{E} \Big\{ 1_{y^i \leq D} \Big[\min(S, w^{i+1}) - \min(S, w^i) \Big] \Big\}.$$

■

A.8 Proposition 4

Proof. In the two cases, we compare the sample-path profit of a portfolio containing option i_t with a portfolio without it. Let $x > 0$ be the amount of option i_t in a given portfolio P_0.

In case (i), replace the x units of option i_t by x units of option k_t. This forms portfolio P_1. For every possible outcome ω, let $q(\omega) \geq 0$ be the optimal amount of option i_t executed when we use portfolio P_0. The cost of option i_t is thus, when outcome ω happens,

$$v_t^{i_t} x + w_t^{i_t} q(\omega) = v_t^{i_t}[x - q(\omega)] + (v_t^{i_t} + w_t^{i_t})q(\omega).$$

Consider now using portfolio P_1 together with the same replenishment strategy as in P_0 with one exception. For option k_t, when outcome ω is realized, we execute, in addition to what was executed when the manufacturer held portfolio P_0, $q(\omega)$ units from the additional x units of capacity. The cost associated with this modification is

$$v_t^{k_t} x + w_t^{k_t} q(\omega) = v_t^{k_t}[x - q(\omega)] + (v_t^{k_t} + w_t^{k_t})q(\omega).$$

Since $0 \leq q(\omega) \leq x$ always, the assumption in the proposition clearly implies that portfolio P_1 with a given replenishment policy yields a smaller cost than portfolio P_0 with its optimal replenishment policy for every outcome ω. Also, since $v_t^{i_t} + w_t^{i_t} > v_t^{k_t} + w_t^{k_t}$ and $v_t^{i_t} > v_t^{k_t}$ and $x > 0$, the difference can never be equal to 0. This implies that P_1 provides a strictly larger expected profit that P_0, and hence option i_t can be excluded from an optimal portfolio.

In case (ii), replace the x units of option i_t by λx units of option j_t and $(1-\lambda)x$ units of option k_t, where $0 < \lambda < 1$ is defined as

$$\lambda = \frac{w_t^{k_t} - w_t^{i_t}}{w_t^{k_t} - w_t^{j_t}}.$$

This forms portfolio P_2. Similarly to the previous case, consider, for portfolio P_2, the same replenishment policy as for P_0 except that, for options j_t and k_t, when outcome ω is realized, we execute, in addition to what was executed when the manufacturer held portfolio P_0, $\lambda q(\omega)$ units from option j_t and $(1-\lambda)q(\omega)$ units from option k_t. The cost is in this case

$$[\lambda v_t^{j_t} + (1-\lambda)v_t^{k_t}][x - q(\omega)] + [\lambda(v_t^{j_t} + w_t^{j_t}) + (1-\lambda)(v_t^{k_t} + w_t^{k_t})]q(\omega).$$

Similarly to case (i), we have that $\lambda v_t^{j_t} + (1-\lambda)v_t^{k_t} < v_t^{i_t}$ and, since $w_t^{i_t} = \lambda w_t^{j_t} + (1-\lambda)w_t^{k_t}$, $\lambda(v_t^{j_t} + w_t^{j_t}) + (1-\lambda)(v_t^{k_t} + w_t^{k_t}) < v_t^{i_t} + w_t^{i_t}$. Hence, using $x > 0$, we see that the cost of P_2 is strictly smaller than the one of P_0, for all outcomes. This implies that we can exclude option i_t from an optimal portfolio. ■

A.9 Proposition 5

Proof. To prove the proposition, assume that the portfolio contains $x > 0$ units of option i_t. We look at the marginal benefit of decreasing x, the amount of option i_t purchased. For every outcome ω (ω describes (D_t, S_t)), let $m_t(\omega)$ be an optimal production price, i.e. a price such that the optimal replenishment policy maximizes

$$m_t(\omega)z - C_t(S_t, D_t, z)$$

where $C_t(S_t, D_t, \cdot)$ is defined in Equation (6). $m_t(\omega)$ can be interpreted in the differentiable case as the optimal dual multiplier in the state corresponding to the events described by ω, i.e.

$$m_t(\omega) = \frac{dU_{t+1}}{dI_{t+1}}(\omega, I_t - d_t + z_t^*) = \frac{dC_t}{dz_t}(\omega, z_t^*).$$

Clearly, since the spot market is uncapacitated, $m_t(\omega) \le S_t$, otherwise we could obtain an infinite profit for some outcome ω.

For an event ω, the marginal benefit of option i_t is equal to

- $m_t(\omega) - w_t^{i_t} - v_t^{i_t}$ if option i_t is executed fully, i.e. $q_t^{i_t}(\omega) = x$;
- $-v_t^{i_t}$ if option i_t is executed partially, i.e. $0 < q_t^{i_t}(\omega) < x$, or it is not executed, i.e. $q_t^{i_t}(\omega) = 0$.

Thus, the marginal benefit of increasing x is equal to $\max\left[m_t(\omega) - w_t^{i_t}, 0\right] - v_t^{i_t} \le [S_t - w_t^{i_t}]^+ - v_t^{i_t}$. After taking expectation, the marginal benefit is less than $\mathbb{E}[S_t - w_t^{i_t}]^+ - v_t^{i_t}$. Since by assumption this is non-positive, we can decrease x to 0 while improving the expected profit. ■

A.10 Theorem 4

Proof. We prove the theorem in the twice-differentiable case. The proof can be easily extended to the general case. When $U_{t+1}(\Phi_t, \cdot)$ is differentiable at I_{t+1}^* and $C_t(\Phi_t, \cdot)$ is differentiable at $z_t^* = I_{t+1}^* - I_t + d_t$, we have that

$$\frac{dU_{t+1}}{dI_{t+1}}(\Phi_t, I_{t+1}^*) = \frac{dC_t}{dz_t}(\Phi_t, z_t^*). \tag{8}$$

This equation can be differentiated with respect to $x_{t'}^k$.

Clearly, when $t' < t$, neither $U_{t+1}(\Phi_t, \cdot)$ nor $C_t(\Phi_t, \cdot)$ depend on $x_{t'}^k$, and therefore, the optimal replenishment policy I_{t+1}^*, as a function of I_t, is independent of $x_{t'}^k$. This proves the first case of the result.

When $t' = t$, $U_{t+1}(\Phi_t, \cdot)$ is independent of $x_{t'}^k$ so we have that

$$\frac{d^2U_{t+1}}{dI_{t+1}^2}(\Phi_t, I_{t+1}^*)\frac{dI_{t+1}}{dx_t^k} = \frac{d^2C_t}{dz_t^2}(\Phi_t, I_{t+1}^* - I_t + d_t)\frac{dI_{t+1}}{dx_t^k} + \frac{d^2C_t}{dz_t dx_t^k}(\Phi_t, I_{t+1}^* - I_t + d_t). \tag{9}$$

Since

$$\frac{dC_t}{dx_t^k}(\Phi_t, z_t) = -\left[\frac{dC_t}{dz_t}(\Phi_t, z_t) - w_t^k\right]^+,$$

we have that

$$\frac{d^2C_t}{dz_t dx_t^k}(\Phi_t, I_{t+1} - I_t + d_t) \le 0.$$

This, together with the fact that $U_{t+1}(\Phi_t, \cdot)$ is concave, $C_t(\Phi_t, \cdot)$ convex, and Equation (9), implies that

$$\frac{dI_{t+1}}{dx_t^k} \ge 0$$

and hence the base-stock level b_t^i increases.

When $t' > t$, since $C_t(\Phi_t, \cdot)$ is independent of $x_{t'}^k$, differentiating Equation (8) yields

$$\frac{d^2U_{t+1}}{dI_{t+1}^2}(\Phi_t, I_{t+1}^*)\frac{dI_{t+1}}{dx_{t'}^k} + \frac{d^2U_{t+1}}{dI_{t+1}dx_{t'}^k}(\Phi_t, I_{t+1}^*) = \frac{d^2C_t}{dz_t^2}(\Phi_t, I_{t+1}^* - I_t + d_t)\frac{dI_{t+1}}{dx_{t'}^k}. \tag{10}$$

We prove this case by induction. Consider $t' = t+1$. By differentiating $U_t(\Phi_{t-1}, \cdot)$, see Equation (3), we have that

$$\begin{aligned}\frac{dU_t}{dI_t}(\Phi_{t-1}, I_t) &= \mathbb{E}_{\Phi_t|\Phi_{t-1}}\frac{d}{dI_t}\Big\{p_t D_t - h_t(I_t) + \max_{I_{t+1}}\left[U_{t+1}(\Phi_t, I_{t+1}) - C_t(\Phi_t, I_{t+1} - I_t + D_t)\right]\Big\} \\ &= -\frac{dh_t}{dI_t} + \mathbb{E}_{\Phi_t|\Phi_{t-1}}\Big[\frac{dC_t}{dz_t}(\Phi_t, I_{t+1}^* - I_t + D_t)\Big].\end{aligned}$$

After differentiating with respect to x_t^k, and using Equation (9), we establish that

$$\begin{aligned}\frac{d^2U_t}{dI_t dx_t^k}(\Phi_{t-1}, I_t) &= \mathbb{E}\Big[\frac{d^2C_t}{dz_t dx_t^k}(\Phi_t, I_{t+1}^* - I_t + D_t) + \frac{d^2C_t}{dz_t^2}(\Phi_t, I_{t+1}^* - I_t + D_t)\frac{dI_{t+1}}{dx_t^k}\Big] \\ &= \mathbb{E}\Big[\frac{d^2U_{t+1}}{dI_{t+1}^2}(\Phi_t, I_{t+1}^*)\frac{dI_{t+1}}{dx_t^k}\Big] \leq 0.\end{aligned}$$

This, together with Equation (10), implies $\dfrac{dI_t}{dx_t^k} \leq 0$.

We can thus initiate an induction to show that for all $m \geq 1$, we have

$$\frac{d^2U_{t+1-m}}{dI_{t+1-m}dx_t^k}(\Phi_{t-m}, I_{t+1-m}) \leq 0 \text{ and } \frac{dI_{t+1-m}}{dx_t^k} \leq 0.$$

For this purpose, we again differentiate $U_{t+1-m}(\Phi_{t-m}, \cdot)$, for $m > 1$, to get,

$$\begin{aligned}\frac{dU_{t+1-m}}{dI_{t+1-m}}(\Phi_{t-m}, I_{t+1-m}) &= \mathbb{E}\frac{d}{dI_{t+1-m}}\left\{\begin{array}{l} p_{t+1-m}D_{t+1-m} - h_{t+1-m}(I_{t+1-m}) \\ + \max\limits_{I_{t+2-m}}\left[\begin{array}{c} U_{t+2-m}(\Phi_{t+1-m}, I_{t+2-m}) \\ -C_{t+1-m}(\Phi_{t+1-m}, I_{t+2-m} - I_{t+1-m} + D_{t+1-m}) \end{array}\right]\end{array}\right\} \\ &= -\frac{dh_{t+1-m}}{dI_{t+1-m}} + \mathbb{E}\Big[\frac{dC_{t+1-m}}{dz_{t+1-m}}(\Phi_{t+1-m}, I_{t+2-m}^* - I_{t+1-m} + D_{t+1-m})\Big].\end{aligned}$$

This implies

$$\frac{d^2U_{t+1-m}}{dI_{t+1-m}dx_t^k}(\Phi_{t-m}, I_{t+1-m}) = \mathbb{E}\Big[\frac{d^2C_{t+1-m}}{dz_{t+1-m}^2}(\Phi_{t+1-m}, I_{t+2-m}^* - I_{t+1-m} + D_{t+1-m})\frac{dI_{t+2-m}}{dx_t^k}\Big] \leq 0.$$

where we used the induction hypothesis for $m-1$ and the convexity of $C_{t+1-m}(\cdot)$. This, by Equation (10), implies that $\dfrac{dI_{t+1-m}}{dx_t^k} \leq 0$, thus concluding the induction. Hence the base-stock level b_t^i decreases as a function of $x_{t'}^k$ for every $t' > t$. ■