Efficient Allocation of Online Advertising Resources

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– Extended Abstract –

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We provide a general and at the same time simple mechanism for the allocation of divisible online advertising resources to a number of heterogeneous firms. The mechanism is ex-post efficient and dominant-strategy incentive compatible. Firms can have multidimensional pieces of private information that may be correlated. The mechanism is robust in the sense that equilibrium bid functions do not depend on any assumptions about other players except for the monotonicity of their otherwise arbitrary payoff functions. Transfers depend on posted price schedules and discounts, which are given based on the average bid function of the other firms evaluated at the fraction of the object not allocated to them. The mechanism can be implemented and run in real-time environments.

Keywords: Auctions, mechanism design, online advertising, paid referrals, search ranks.

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1 Introduction

There is a plethora of available online advertising resources, such as paid placements in the form of banner ads placed on a company’s web pages, or the targeted display of advertising content as a result of more detailed consumer interaction, at the end of a search, while playing online games, or when checking email, to name just a few. For simplicity, we refer to a company having advertising resources at its disposal as an intermediary, since it acts as a link between firms who are willing to pay for these resources and consumers whose purchasing decisions the advertising firms seek to influence in their favor. Such intermediaries could be electronic portals such as AOL or Yahoo, search intermediaries such as AltaVista, or electronic newspapers such as CNN.com. In that sense, the intermediary is the link between the two sides of a two-sided market with consumers on one side and advertising firms on the other (Rochet and Tirole, 2003). We distinguish online advertising resources from offline advertising resources, since the former can be fundamentally considered divisible while the latter in most cases are not. For instance, the New York Times print edition can allocate a particular advertising spot each day only to exactly one advertiser. Similarly, radio and TV commercials are broadcast to all listeners or viewers simultaneously, which makes these offline advertising resource indivisible. Online advertising resources, on the other hand, can be point-cast or randomized, and thus divided between bidding firms at almost arbitrary proportions. For instance, a critical front page ad at the Wall Street Journal’s online edition can be allocated in part to one firm and in part to another firm by splitting the number of impressions for each ad corresponding to the two firms’ payments to the intermediary.

The mechanisms currently employed by intermediaries to allocate advertising resources vary by the nature of the resource. Search ranks are usually auctioned off, for instance using variants of the generalized second price auction (Edelman et al., 2005) in which advertisers announce their willingness to pay per click on their advertisement. The search engines then display the results in a sequence that depends on these bids and the relevance to the searched phrase and charge the advertisers the second highest bid per click. Other target advertising is sold off in chunks on a pay-per-click or pay-per-impression basis, and sometimes even on a ‘pay-for-performance’ basis as a percentage of sales. There are also intermediaries that act as advertising demand aggregators, such as Advertising.com or Doubleclick.com, who resell advertising resources from other intermediaries. In this case, we restrict attention to either one of the intermediaries and its decision to allocate its available resources, after the opportunity cost for them has been sunk. For instance, with our model we may address Advertising.com’s decision about allocating its advertising resources to firms after having obtained a large number of ad impressions from Yahoo, perhaps through a negotiated agreement.

We now outline three important shortcomings in existing mechanisms:

First, implicit in the design of many a commercially used advertising allocation mechanism is that the advertising resource in question is allocated to one of \( N \) interested firms through some form of competitive bidding process. This bid typically consists of a scalar value, such as the bidding firm’s willingness to pay for a rank-one search referral. In some instances, the bid may have additional components, such as the maximum available budget, but usually, the information that the intermediary receives about the value of the resource is limited to a point, which for a divisible resource implies constant returns to scale, at least over a range (e.g., until the budget constraint becomes binding). It thus also implies a comparability among the firms of the returns to the advertising resource in question. In reality, the value for advertising resources is likely to be nonlinear in the allocation and heterogeneous across firms. For instance, due to word of mouth

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1 There might be the possibility of allocating a particular page in a local section of the newspaper for different local editions, which does not fundamentally alter the indivisible nature of the print resource.
effects and usage externalities, one firm might have increasing returns to a resource, while another firm may experience decreasing returns to scale. An appropriate allocation mechanism should therefore be able to deal with demand functions, rather than point valuations.

Second, some existing allocation mechanisms (e.g., the generalized second-price auction mentioned earlier) are predicated upon the assumption that there is a common preference ranking among the advertising resources, such as search ranks. However, in general there is no reason for this to be true: for instance, on a per-click basis, since the first search link is often followed by low-value consumers, the most valuable search traffic is more likely to come from lower search ranks than from the first one. In general, the preference ordering over \( K \) different available resources (e.g., search ranks 1 through \( K \), or banner ads on pages 1 through \( K \)) might vary across bidding firms.

Third, the mechanism should be robust in the sense that equilibrium-bidding behavior should not depend heavily on common-knowledge assumptions\(^2\) and implementation should therefore be in dominant strategies and lead to truthful bidding behavior (e.g., the generalized second-price auction is not truthful). It should also be robust in the sense that it is easy to implement and its complexity does not increase substantially by adding more real-world features.

Research Question. In this paper we construct a dominant-strategy incentive compatible mechanism for the allocation of online advertising resources that (1) allows for heterogeneous bidders with multidimensional private type information which may be jointly or privately distributed, (2) allows the allocation of \( K \) divisible resources that are jointly valued by each firm (e.g., there may be complementarities or anti-complementarities between resources from the perspective of any bidding firm, and (3) may lead to ex-post individual rationality (on the part of the firms), and, more importantly, to an ex-post efficient allocation. The equilibrium bids are in dominant strategies and thus do not depend on any distributional assumptions; they do not depend on the number of bidders, and also do not require any knowledge by any one firm about the parametric assumptions about another firm’s payoff function. Since it is well known (Green and Laffont, 1977) that there cannot exist allocation mechanisms (in the presence of asymmetric information) that at the same time are ex-post efficient, dominant-strategy incentive compatible, and balance the budget, our mechanism is in general not budget-balancing. We discuss the impact of this on the intermediary’s equilibrium revenues.

2 The Basic Setup

We consider \( N \) heterogeneous firms that have nonnegative valuations for fractional allocations of \( K \) available search ranks provided by a search intermediary. The ranks might be for the same keywords or they might range across different keywords. The fractional allocation \( x^k_i \) of rank \( k \) to firm \( i \in \{1, ..., N\} \) is nonnegative (with \( N \geq 2 \)) and each rank is fully assigned, i.e., \( \sum_{i=1}^{N} x^k_i = 1 \) for all \( k \in \{1, ..., K\} \). Firm \( i \)'s utility for its rank allocation \( x_i = (x^1_i, ..., x^K_i) \) is denoted by \( v_i(x_i; \theta_i) \), where \( \theta_i \in \Theta_i \) represents firm \( i \)'s private information, or ‘type,’ as an element of its type space \( \Theta_i \) which is a bounded measurable subset of some finite-dimensional Euclidean space. The function \( v_i : [0,1]^K \times \Theta_i \to \mathbb{R} \) is assumed to be smooth and increasing in its first argument. We assume that the firms’ type vector \( \theta = (\theta_1, ..., \theta_N) \) is distributed with the joint cdf \( F : \Theta \to [0,1] \), where \( \Theta = \Theta_1 \times \cdots \times \Theta_N \) is the joint type space. Before the beginning of the game each firm \( i \) observes its type \( \theta_i \) privately; its beliefs about the other firms’ types \( \theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_N) \) are thus distributed according to the conditional distribution function \( F(\theta_{-i} | \theta_i) \), which is obtained via Bayesian updating. The sequence of events is as follows. First the intermediary announces and

\(^2\) This is sometimes also referred to as the ‘Wilson doctrine’ (Wilson, 1987).
commits to its mechanism, which includes a fixed posted $K$-dimensional price vector $p = (p^1, ..., p^K)$, the components of which correspond to the different available resources. Each firm $i$, conditional on having observed its type $\theta_i$, then submits to the intermediary a real-valued bid function $b_i(\cdot; \theta_i)$, defined on $[0,1]^K$, which (after all bid functions have been collected) is used to determine the firms’ allocation matrix $X = [x^k_i]_{i,k=1}^{N,K}$ as well as their transfer payments. A peculiar feature of our mechanism is that the firms’ bids actually act as discounts that will be subtracted from the payment implied by the posted price schedule. Each firm $i$ makes a monetary transfer of $t_i$ to the intermediary which depends on $X$ and all other firms’ bid functions. The precise dependence of the fractional allocations and monetary transfers on the firms’ bid functions and the intermediary’s payoffs is specified by the intermediary’s allocation mechanism.

### 3 The Allocation Mechanism

The intermediary’s mechanism $M$ maps the firms’ bid-function vector $b = (b_1, ..., b_N)$ to an allocation matrix $X$ and a corresponding vector of monetary transfers $t = (t_1, ..., t_N)$, such that

$$t_i(b, X) = p \cdot x_i - \frac{1}{N-1} \sum_{j \neq i} b_j(\bar{x}-j; \theta_j),$$

where $\bar{x} = x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_N$ is the vector containing the respective sums of fractional rank-allocations to firms other than firm $i$. Note that an average over the other firms’ bids defines the discount that firm $i$ obtains off the total payment $p \cdot x_i = \sum_{k=1}^{K} p^k x_i^k$ which is implied by the posted price vector $p$. Each firm $j$’s bid function is evaluated not only at $i$’s allocation, but at the sum of all allocations other than $j$’s (which includes $i$’s allocation). The allocation matrix $X$ solves the intermediary’s profit maximization problem. The intermediary, by maximizing its profits, trades off discounts versus revenues, subject to the constraints on the availability of online advertising resources.

**Proposition 1.** Given the intermediary’s mechanism $M$, the firms’ bidding equilibrium is in dominant strategies. For any $i \in \{1, ..., N\}$ firm $i$’s equilibrium bid function is given by

$$b_i(x_i; \theta_i) = v_i(1; \theta_i) - v_i(1 - x_i; \theta_i) \geq 0$$

for all $x_i \in [0,1]^K$ and all $\theta_i \in \Theta_i$. 


Each firm $i$'s equilibrium bid function in Proposition 1 only depends on firm $i$'s gross payoff function $v_i(\cdot; \theta_i)$ which is naturally part of firm $i$'s private information. In particular it is independent of the distributional assumptions about the other firms' types. The intuition for this somewhat surprising result is as follows. Provided truthful bidding, the intermediary, who collects the bid functions from all the firms, can make a proper allocation decision which reflects the full realization of the type vector $\theta \in \Theta$. Thus, anticipating this each firm $i$ can now use the intermediary’s optimality conditions as a way of inferring system-wide marginal cost corresponding to the Lagrange multipliers associated with common constraints (such as $x_1 + \cdots + x_N = 1$) from its own gross payoff function. Note that for this to work, it is crucial that firm $i$’s payoffs are independent (at least before the intermediary optimizes) of its allocation and bidding function. In addition, the mechanism $M$ provides nice aggregation properties; for any feasible allocation matrix $X$ it needs to be true that $b_j(\bar{x}; \theta_j) = b_j(1 - x_j; \theta_j)$, which, if the bidding function in Proposition 1 is substituted, provides incentives for the intermediary to maximize the sum of all firms' payoffs and thus implement an ex-post efficient allocation of advertising resources. Define $W(X, \theta) \equiv \sum_{j=1}^N v_j(x_j; \theta_j)$.

Then, the following result gives a more precise meaning to this intuitive argument.

**Proposition 2.** For any $\theta \in \Theta$ the intermediary’s equilibrium payoff is given by $W(X, \theta) - (W(1, \theta) - \bar{p})$, where $\bar{p} = p^1 + \cdots + p^K$ and $1$ is an $N \times K$-matrix of ones.

Since $W(1, \theta)$ is not attainable by any feasible allocation (unless all firms but one have gross payoffs that are identically equal to zero), it becomes clear that the intermediary is unable to extract all the surplus from the agent. In fact, as becomes clear later, by increasing the posted price, participation in the intermediary’s mechanism will drop (eventually to zero for $p$ large enough). Hence there exists an interior optimal posted price that maximizes the intermediary’s expected revenue. Clearly, this revenue maximization would need to depend on the intermediary’s very own beliefs about the distribution of the firms’ types.

**Proposition 3.** Given any type vector $\theta \in \Theta$, under mechanism $M$ any positive equilibrium allocation matrix $X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ is determined by $v'_i(x_i; \theta_i) = v'_1(x_1; \theta_1)$ for all $i \in \{2, \ldots, N\}$ and $x_1 + \cdots + x_N = 1$.

The reason for restricting the scope of Proposition 3 to equilibrium allocation matrices with positive entries, i.e., where no firm receives zero payoffs, is that then there is only a single equality constraint in the entire equilibrium problem, the Lagrange multiplier of which can be eliminated by setting the respective firms’ marginal payoffs at the optimal allocation equal. This implies efficiency.

**Corollary 1.** Under mechanism $M$ any positive equilibrium allocation matrix $X$ is efficient.

A sufficient condition for interiority (i.e., positivity) of the equilibrium allocation matrix is that the marginal change of the firms’ gross payoff functions is sufficiently close to zero allocations. If the slopes of all firms’ payoffs remain finite within the set of possible allocations $[0, 1]^K$, then this is also sufficient. The following corollary of Proposition 3 uses a slightly stronger requirement as a sufficient condition for a somewhat stronger result.

**Corollary 2.** Let $\theta \in \Theta$ and assume that the firms’ payoff functions $v_1(\cdot; \theta_1), \ldots, v_N(\cdot; \theta_N)$ are concave in the fractional allocations they receive. If every entry of $v'_i(0, \theta_i) > 0$ is large enough (e.g., ensured by the Inada-type conditions $\frac{\partial v'_i}{\partial x_i}(0, \theta_i) = \infty, k \in \{1, \ldots, K\}$) for all $i \in \{1, \ldots, N\}$, then there exists a unique positive equilibrium allocation matrix $X = G(\theta)$. 


4 The Intermediary’s Revenue Maximization Problem

From the results in the last section we conclude that mechanism M is dominant-strategy incentive compatible and ex-post efficient. Clearly, this implies that it cannot at the same time be budget balanced and individually rational. Thus, the intermediary faces a choice: (i) to sufficiently subsidize the mechanism, ensuring that firms always participate (“social objective”), i.e., set the posted price vector \( p \) to zero; or (ii) optimize the posted price vector \( p \) so as to maximize revenues (“capital objective”).

**Social Objective.** By charging a zero posted price \( (p = 0) \) the intermediary can ensure full participation: any firm obtains at least zero and is therefore at least indifferent between participating or not. However, since the intermediary initially commits to the mechanism, its discounts to some firms may exceed their payments, so that the intermediary ends up out of pocket. Depending on the intermediary’s beliefs about the distribution of the firms’ types, this implies a lowest possible price that the intermediary would need to charge in order for the mechanism M to be ex-ante willing to run it. This would exclude some types from participating, but the mechanism still implements an efficient outcome.

**Capital Objective.** The intermediary’s choice of the price vector \( p \) determines the types of firms that would find it ex-post individually rational to participate in the auction. The intermediary anticipates this and chooses a \( \bar{p} \) to maximize its expected total transfer conditional on the fact that the set of participating types ("participation set") is potentially a strict subset of the type space.

5 Discussion

The allocation of an intermediary’s online advertising resources generally takes place in a fluid environment with a nonstationary population of potential advertisers. As a result the private payoff functions of the bidding firms can be expected to depend on a large number of unknown parameters. Our approach essentially does not depend on the shape of the firms’ payoff functions and requires nothing but smoothness and monotonicity in the allocation and measurability in the (possibly multidimensional) parameters. As mentioned in the Introduction, the key feature that distinguishes the allocation of online advertising resources from other (offline) advertising resources, is its almost seamless divisibility, which can be implemented virtually without resorting to intertemporal randomization (which, strictly speaking, would need to be fully analyzed using a dynamic modeling framework). Our mechanism robustly implements an efficient allocation for such divisible goods, which to the best of our knowledge has not been accomplished so far in neither the information systems nor the economics literature.

Upon first sight, the very fact that this mechanism asks for the equivalent of a full payoff function from firms might seem to add substantial complexity over a standard allocation method where firms announce single or multi-component bids. However, the complexity is relatively small for the intermediary, who has to solve a simple constrained optimization problem using whatever bid functions have been supplied to him by the firms. Therefore, if some firms prefer reduced bidding complexity, they are free to simplify the structure of their bid functions arbitrarily if they are willing to compromise somewhat on their surplus. For instance, they could just submit a linear payoff function, which would be the equivalent to a standard bid in a standard-auction environment. If its payoff function contains a maximum value, then that is equivalent to a budget constraint. In this vein, the intermediary could offer potential advertisers a simple menu of piecewise linear payoff functions, which would limit bids to a simple vector of breakpoints and slopes. Such bid vectors could be processed very fast and the solution be obtained by solving a linear program, which can be processed fast enough to provide a scheme that can be applied in real-time applications.